ON QUIVER REPRESENTATIONS AND DYNKIN DIAGRAMS

Nikita Borisov

A THESIS

in

Mathematics

Presented to the Faculties of the University of Pennsylvania

in

Partial Fulfillment of the Requirements for the

Degree of Master of the Arts

2024

Supervisor of Thesis

Wolfgang Ziller, Professor of Mathematics Emeritus

Graduate Group Chairperson

Ron Donagi, Thomas A. Scott Professor of Mathematics

ACKNOWLEDGEMENT

I would like to thank Wolfgang Ziller for his course on Lie groups. I would also like to thank Emmett Lennen for his insightful comments on the wild problem and relevant references.

ABSTRACT

ON QUIVER REPRESENTATIONS AND DYNKIN DIAGRAMS

Nikita Borisov

Wolfgang Ziller

In this expository paper, we develop the theory of root systems and study them using Dynkin diagrams. We then develop the the theory of quiver representations and prove a connection between these representations and root systems, called Gabriel's theorem, which states that if a quiver arose from a Dynkin diagram then its indecomposable quiver representations are counted by positive roots. The theory is then generalized beyond quivers arising from Dynkin diagram, which have connections to Kac-Moody Lie algebras. We also give applications to the representation theory of finite dimensional $\mathbb{C}$-algebras and classifying finite subgroups of $SU(2)$. We follow [6],[8] for material on root systems and Lie algebras and [2],[5] for the theory of quiver representations.

TABLE OF CONTENTS

CHAPTER 1

Introduction

Semisimple Lie algebras were first classified by Killing and Cartan. Dynkin then proposed an invariant defined from the root system of a semisimple Lie algebra, called the Dynkin diagram. This gave a simpler presentation of the classification as well as capturing certain properties of the Lie algebras. Similar diagrams had previously been studied by Coxeter and Witt in connection to root systems and the classification of reflection groups.

Dynkin diagrams have since appeared in many areas of math. A subset of the diagrams, called simply-laced Dynkin diagrams, has connections to classifying finite subgroups of $SU(2)$, spectral graph theory, generalized quadrangles (a type of incidence structure), and catastrophes in bifurcation theory. The main focus of this expository paper will be the connection between simply-laced Dynkin diagrams and the theory of quiver representations. A quiver is a directed multi-graph and a quiver representation attaches a linear space to each vertex and a linear map to each edge. Gabriel discovered that the quivers which admit only finitely many indecomposable representations, up to isomorphism, are precisely the simply-laced Dynkin diagrams. Furthermore, there is a one-to-one correspondence between these representations and positive roots of the associated root system. The theory of classifying indecomposable representations for non-simply-laced quivers is more complicated and was studied by Kac.

This paper is meant to be self-contained and only assumes familiarity with the standard undergraduate mathematics curriculum. We start chapter 2 with an axiomatic presentation of root systems and define Dynkin diagrams as invariants associated with root systems following [6],[8]. We briefly describe how Dynkin diagrams classify semisimple Lie algebras, although this material will not be necessary for the latter portions of the paper. In chapter 3, we provide an introduction to quiver representations and present the problem of classifying indecomposable representations. In chapter 4, we prove Gabriel's theorem following [2],[5] by using reflection functors and insights from spectral graph theory. In chapter 5, we discuss generalizations of the theory to other quivers and provide applications in the representation theory of algebras and the classification of finite subgroups of $SU(2)$.

CHAPTER 2

Root systems, Dynkin diagrams, and semisimple Lie algebras

We will start by introducing root systems and the classification of their Dynkin diagrams. We will also introduce extended Dynkin diagrams and the Weyl group. Then we will give a brief description of how Dynkin diagrams classify semisimple Lie algebras. We mostly follow the notes [6] and chapter 4 of [8].

2.1. Root systems and Dynkin diagrams

We adopt the following definition from [6].

Definition 2.1.1. *A **root system** is a finite subset, Δ, of a real inner product space $(V, \langle , \rangle)$, such that:*

- *(a) Δ spans V and $0 \notin \Delta$*
- *(b) $\frac{2\langle \alpha,\beta \rangle}{\langle \alpha,\alpha \rangle} \in \mathbb{Z}$ for $\alpha, \beta \in \Delta$*
- *(c) If $\alpha, \beta \in \Delta$ and $\beta = c\alpha$, then $c = \pm 1$*
- *(d) If $\alpha, \beta \in \Delta$, then $\beta - \frac{2\langle \alpha,\beta \rangle}{\langle \alpha,\alpha \rangle}\alpha \in \Delta$. Notice that if $\alpha \in \Delta$, so is $-\alpha$.*

The elements of Δ are called roots. Given $\alpha, \beta \in \Delta$, the integers, $n_{\alpha,\beta} := \frac{2\langle \alpha,\beta \rangle}{\langle \alpha,\alpha \rangle}$, are called **Cartan integers**. Since

$$n_{\alpha,\beta} \cdot n_{\beta,\alpha} = \frac{4\langle \alpha, \beta \rangle^2}{|\alpha|^2 \cdot |\beta|^2} = 4\cos^2(\angle(\alpha, \beta)) \tag{2.1}$$

is an integer, the only possible values for the angle between α and β are 0, π, $\pi/2$, $\pi/3$, $2\pi/3$, $\pi/4$, $3\pi/4$, $\pi/6$, $5\pi/6$. Furthermore, if α, β are not linearly dependent, then $\cos^2(\angle(\alpha, \beta)) < 1$ and the only possible values for $n_{\alpha,\beta}$ are $\pm 1, \pm 2, \pm 3$.

Also observe that for $\alpha, \beta \in V$, $\sigma_\alpha(\beta) := \beta - \frac{2\langle \alpha,\beta \rangle}{\langle \alpha,\alpha \rangle}\alpha$ is the reflection of β through the hyperplane, $\alpha^\perp$.

Example 2.1.2. Consider the set $\Delta = \{\pm(e_i - e_j) : 1 \leq i < j \leq n+1\} \subseteq \mathbb{R}^{n+1}$, for $n \geq 1$. One can check that all the properties of a root system are satisfied. We will call this root system, A_n. |

Example 2.1.3. Consider the set $\Delta = \{\pm(e_i + e_j), \pm(e_i - e_j), \pm e_i : 1 \leq i < j \leq n\} \subseteq \mathbb{R}^n$, which will also turn out to be a root system which we call B_n. Notice that unlike the roots of A_n, these root vectors can have different lengths. |

Since roots always come in positive-negative pairs, $\pm\alpha \in \Delta$, we wish to partition Δ into a set of positive and negative roots. This is always possible, but there is no canonical way to do this. It will turn out the regardless of the choice, the same invariants will arise.

Definition 2.1.4. *Given a root system $\Delta \subseteq V$, fix a $v \in V$ such that $\langle v, \alpha \rangle \neq 0$ for all $\alpha \in V$ (which exists since Δ is finite). Then the* ***positive roots*** *with respect to v are the elements of*

$$\Delta^+ := \{\alpha \in \Delta : \langle v, \alpha \rangle > 0\},$$

and the set of ***negative roots*** *is defined analogously, $\Delta^- = -\Delta^+$.*

A positive root $\alpha \in \Delta^+$ is called ***simple*** *if it is not a sum of two other positive roots. The set of simple roots will be denoted Δ_s. The cardinality of Δ_s is called the* ***rank*** *of Δ.*

If r is rank of Δ, then the ***Cartan matrix*** *of Δ is the r-by-r matrix $C := (n_{\alpha,\beta})_{\alpha,\beta \in \Delta_s}$*

We claim that the rank is independent of choice of positive roots and so is the Cartan matrix (up to reordering of simple roots), justifying the definition. See [8] for the proof of this statement.

Example 2.1.5. In Example 2.1.2, by choosing v with $v_1 > v_2 > \cdots > v_{n+1} > 0$ for all i, we get $\Delta^+ = \{e_i - e_j : i < j\}$ and the simple roots are $\{e_i - e_{i+1} : 1 \leq i \leq n\}$. Hence the rank is n. The

Cartan matrix will have entries

$$C_{i,j} = \begin{cases} 2, & \text{if } i = j \\ -1, & \text{if } |i-j| = 1 \\ 0, & \text{otherwise} \end{cases} \cdot$$

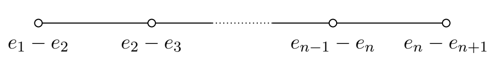
Defining v the same way for B_n, yields $\Delta^+ = \{e_i - e_j, e_i + e_i, e_i : 1 \le i < j \le n\}$ and $\Delta_s = \{e_1 - e_2, \ldots, e_{n-1} - e_n, e_n\}$. Notice that the rank in both cases is the subscript n.

We can now define the Dynkin diagram of Δ. It will require a choice of positive (and hence simple roots) and we again refer to [8] to see that the Dynkin diagram is independent of this choice.

Definition 2.1.6. *Given a root system $\Delta \subseteq V$, the* ***Dynkin diagram*** *is a multi-graph, $\Gamma = \Gamma(\Delta)$, with potentially oriented edges. The vertices of Γ are indexed by Δ_s. Given distinct $\alpha, \beta \in \Delta_s$, we draw $n_{\alpha,\beta} \cdot n_{\beta,\alpha}$ edges connecting the corresponding vertices. Furthermore, if α, β have different lengths then we put an arrow over the edges pointing from the longer root to the shorter one.*

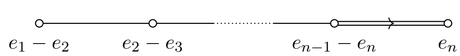
Note that since distinct $\alpha, \beta \in \Delta_s$ are both positive roots, they are linearly independent and so $n_{\alpha,\beta} \cdot n_{\beta,\alpha} = 1, 2, 3$ by (2.1). By analysing (2.1), the roots will have the same lengths in the case when this quantity is 1 and will have different lengths otherwise. A Dynkin diagram with no double or triple edges will just be an ordinary graph, and is referred to as a **simply laced** Dynkin diagram. These will be of greatest importance when considering quiver representations.

Example 2.1.7. The Dynkin diagram of A_n can be read off the Cartan matrix:



,

the path graph with n vertices. Similarly, for B_n with $\Delta_s = \{e_1 - e_2, \ldots, e_{n-1} - e_n, e_n\}$, we get

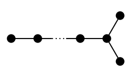
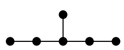
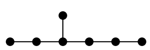
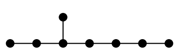


,

where the arrows point right since $e_{n-1} - e_n$ is longer than e_n.

Theorem 2.1.8. *The root system corresponding to a Dynkin diagram is a direct sum of the root systems corresponding to each of its connected components, which are also Dynkin diagrams.*

Any Dynkin diagram arising from a root system is a union of the following Dynkin diagrams with the number of roots indicated after the diagram:

- A_n: $|\Delta| = n(n+1)$
- B_n, $n \geq 2$: $|\Delta| = 2n^2$
- C_n, $n \geq 2$: $|\Delta| = 2n^2$
- D_n, $n \geq 4$:  $|\Delta| = 2n(n-1)$
- E_6:  $|\Delta| = 72$
- E_7:  $|\Delta| = 126$
- E_8:  $|\Delta| = 240$
- F_4: $|\Delta| = 48$
- G_2: $|\Delta| = 12$

In each case, the subscript indicates the number of vertices which is also the rank of the corresponding root system. The Dynkin diagrams E_6, E_7, E_8, F_4, G_2 are called ***exceptional****. The simply laced Dynkin diagrams are thus unions of A_n, D_n, E_n and we call such diagrams* ***ADE-type****.*

We have already seen that A_n, B_n have root systems and we need to exhibit the underlying root systems for the remaining Dynkin diagrams in the above list. The diagram C_n arises from $\Delta = \{\pm(e_i - e_j), \pm(e_i + e_j), \pm 2e_i : 1 \leq i < j \leq n\}$, while D_n arises from $\Delta = \{\pm(e_i - e_j), \pm(e_i + e_j) : 1 \leq i < j \leq n\}$. The exceptional root systems are more complicated, but can be constructed using

Proposition 2.3.2.

We will prove the following weaker theorem

Theorem 2.1.9. *Any simply laced Dynkin diagram is of ADE-type.*

First we will need to better understand simple roots and Cartan matrices.

Lemma 2.1.10. *We have the following*

1. *If $\alpha, \beta \in \Delta_s$ are distinct, then $\langle \alpha, \beta \rangle \leq 0$ and hence $n_{\alpha,\beta} \leq 0$.*

2. *The elements of Δ_s form a basis for W.*

3. *If $\alpha \in \Delta^+$ is written as a linear combination $\alpha = \sum_{x \in \Delta_s} n_x x$, then the $n_x \in \mathbb{N}$.*

Proof. Suppose for contradiction that $\langle \alpha, \beta \rangle > 0$. Then the only possible angles between α, β are $\pi/3, \pi/4, \pi/6$, with the ratios between the root lengths being, $1, \sqrt{2}, \sqrt{3}$, respectively. In each case, we can use property 2.1.1.d to see that $\beta - \alpha \in \Delta$. Then $\beta - \alpha \in \Delta^+$ or its negative, $\alpha - \beta \in \Delta^+$. In the first case $\alpha = \alpha + (\beta - \alpha)$ a sum of two positive roots, contradicting that α was simple. The second case is similar.

Since the span of the Δ_s contains positive roots and hence all roots, they span W. Suppose there were a linear dependence of the Δ_s, $\sum_{x \in \Delta_s} n_x x = 0$. Then we may separate terms with positive and negative coefficients, n_x, yielding $P = \sum_{n_x > 0} n_x x$ and $N = \sum_{n_x < 0} n_x x$. Since $P + N = 0$,

$$\langle P, P \rangle = -\langle P, N \rangle = \sum_{n_x > 0} \sum_{n_y < 0} n_x (-n_y) \langle x, y \rangle \leq 0,$$

since $\langle x, y \rangle \leq 0$ by the first part. Thus, $P = N = 0$. Since positive roots were defined as having positive inner product with some v, we would have $\langle P, v \rangle > 0$ if any of the $n_x > 0$. Similarly, we see that the coefficients of N are zero.

To see the last fact, we just have to write any positive root, α, as a sum of simple roots. If α is not simple, it is a sum of two positive roots, β, γ. Now repeat with this process with β, γ. This process must terminate with a sum of simple roots, since at every level of the algorithm we introduce at least one new positive root and there is a finite number of these. □

Remark 2.1.11. Note that the Dynkin diagram relies only on knowledge of the Cartan matrix. Similarly, given a Dynkin diagram we can recover the Cartan matrix. First notice that $n_{\alpha,\alpha} = 2$, determining the diagonal entries. By Lemma 2.1.10, $n_{\alpha,\beta} = 0$, if α, β are disconnected and $n_{\alpha,\beta} = -1$, if α, β are connected by a single edge in Γ. If they are connected by 2 (resp. 3) edges, with α being the shorter root (this information is recorded by Γ), then $n_{\alpha,\beta} = -2$ (resp. -3) and $n_{\beta,\alpha} = -1$. |

Thus, for simply laced Dynkin diagrams, Γ, the Cartan matrix, $C = 2I - A$, where A is the adjacency matrix of the ordinary graph Γ. and I is the identity. We have the following important fact about Cartan matrices.

Proposition 2.1.12. *The Cartan matrix, C, of a root system is invertible.*

Proof. Consider a vector $v \in \ker C$. Let $\alpha = \sum_{x \in \Delta_s} v_x x$. Then $Cv = 0$ implies

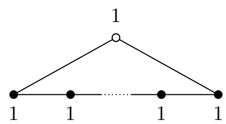
$$0 = \sum_{y \in \Delta_s} n_{x,y} v_y = \frac{2}{\langle x, x \rangle} \sum_{y \in \Delta_s} \langle x, y \rangle v_y = \frac{2}{\langle x, x \rangle} \langle x, \alpha \rangle$$

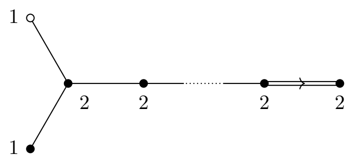
for all $x \in \Delta_s$. But since the simple roots form a basis for W, this forces $\alpha = 0$ and by linear independence of simple roots, $v = 0$. Thus, $\ker C = 0$ and C is invertible. □

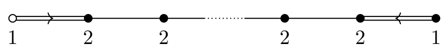
This provides a way to rule out certain diagrams from being Dynkin diagrams (which arise from a root system by definition) by using 2.1.11 to compute the corresponding Cartan matrix and checking if it is invertible.

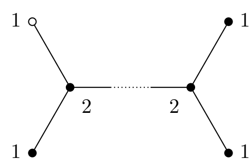
Example 2.1.13. The following diagrams, Γ, cannot be Dynkin diagrams of any root system since the Cartan matrices, C, are not invertible. In particular, we provide a labelling of the vertices $\kappa \in \mathbb{R}^{V(\Gamma)}$ such that $\kappa \in \ker C$. For simply laced Dynkin diagrams the condition $\kappa \in \ker C$ reduces to κ being a 2-eigenvector of Γ, i.e. $2\kappa(x) = \sum_{y\sim x} \kappa(y)$, for all $x \in V(\Gamma)$, where the sum is taken over neighbors of x.

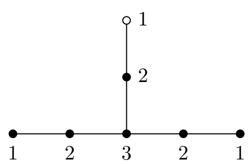
Each of these diagrams, while not being a Dynkin diagram, arise from a Dynkin diagram by adding an additional vertex (denoted by a blank circle) and are called **extended Dynkin diagrams**. In each case, the number of vertices is one more than the subscript

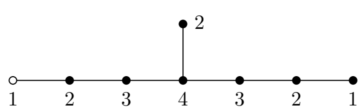
- $\hat{A}_n$, $n \geq 2$: 

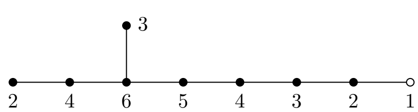
- $\hat{B}_n$, $n \geq 2$: 

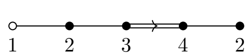
- $\hat{C}_n$, $n \geq 2$: 

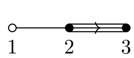
- $\hat{D}_n$, $n \geq 4$: 

- $\hat{E}_6$: 

- $\hat{E}_7$: 

- $\hat{E}_8$: 

- $\hat{F}_4$: 

- $\hat{G}_2$: 

These examples will be crucial in proving the weaker classification theorem 2.1.9. Consider a subset $F \subseteq \Delta_s$ and observe that $\mathbb{R}\langle F\rangle \cap \Delta \subseteq \mathbb{R}\langle F\rangle$ is a root system. Properties 2.1.1a-c clearly still hold and if roots α, β are in the span of F, then so is $\sigma_\alpha(\beta)$, so 2.1.1d holds as well. Thus, *any induced subgraph of a Dynkin diagram is again a Dynkin diagram.*

Hence, a Dynkin diagram cannot contain any of the extended Dynkin diagrams as subgraphs. Theorem 2.1.9 will then follow from the following lemma.

Lemma 2.1.14. *If a connected, simply laced diagram, Γ, does not contain any of $\hat{A}_n, \hat{D}_n, \hat{E}_n$ as an induced subgraph, then it is one of A_n, D_n, E_n.*

Proof. Suppose for contradiction Γ had a cycle. Then there would be a shorted cycle consisting of vertices $x_1, \ldots, x_{n+1}$ and the induced subgraph from these vertices will be $\hat{A}_n$, a contradiction (there will be no edges going “across" the induced subgraph by minimality of the cycle). Therefore, Γ is a tree. Similarly, since Γ does not contain $\hat{D}_4$ as an induced subgraph, all vertices of Γ have degree ≤ 3. Since Γ does not contain $\hat{D}_n$ for $n \geq 5$, at most one vertex may have degree 3.

If no vertex has degree 3 then Γ is forced to be a path graph, A_n, and we are done. Otherwise, assume Γ has exactly one central vertex of degree 3 and three branches. Then let $2 \leq a \leq b \leq c$ denote the lengths of the branches, where the length of a branch is the number of vertices in the branch, including the central one. We must have $a = 2$ or else $a, b, c \geq 2$ and Γ would contain $\hat{E}_6$.

Then if $b = 2$, Γ is of the form D_n and we are done. Otherwise, $b = 3$, since if $b > 3$, Γ would contain $\hat{E}_7$. Since Γ may not contain $\hat{E}_8$, the remaining possible values for $c = 3, 4, 5$, corresponding to Γ being E_6, E_7, E_8, respectively. ☐

Remark 2.1.15. We saw that the simply laced extended Dynkin diagrams have eigenvalue 2 as a graph. It turns out that 2 is also the spectral radius of these graphs (note that graphs always have a real spectra since their adjacency matrices are symmetric). An exercise of Godsil and Royle [3] actually claims that *a connected graph has spectral radius 2, if and only if it is one of the $\hat{A}_n, \hat{D}_n, \hat{E}_n$.*

We refer to a math stack exchange post [7] for a sketch of the proof, which bears some similarities to the proof of Lemma 2.1.14.

Now observe that if Γ' is a proper induced subgraph of Γ, then $A' = A(\Gamma')$ is a sub-matrix of $A = A(\Gamma)$ and hence the spectral radius $\lambda(\Gamma') < \lambda(\Gamma)$. This is because the spectral radius of symmetric A' is the maximum of $|A'x|$ over unit length x. Hence by appending zeros to x, to get $\hat{x}$, we have $|A'x| \leq |A\hat{x}|$. It is a little trickier to see that the spectral radius inequality is strict, but can be done with the same argument.

A consequence is that graphs of ADE-type, Γ, have spectral radius $\lambda(\Gamma) < 2$. By Lemma 2.1.14, if Γ is not ADE-type, then it contains an extended Dynkin diagram as a subgraph and hence has $\lambda(\Gamma) \geq 2$. Therefore, *a connected graph has spectral radius <2, if and only if it one of the* A_n, D_n, E_n. As a result, we also get that any proper subgraph $\hat{A}_n, \hat{D}_n, \hat{E}_n$ is of ADE-type. |

Remark 2.1.16. In [5], it is pointed out that the A_n, D_n, E_n are precisely the graphs with three branches of length $1 \leq a \leq b \leq c$, such that $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} > 1$, where now the path graphs, A_n, are included with $a = 1$. In this context, $\hat{E}_6, \hat{E}_7, \hat{E}_8$ are precisely the three branch graphs with a, b, c satisfying $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$. The parameters a, b, c will make a return in Section 5.3. |

2.2. Extended Dynkin diagrams from maximal roots

In this section, we mention where the extended Dynkin diagrams introduced in Example 2.1.13 come from, i.e. how the additional vertex is connected to the underlying Dynkin diagram. We will also see how labellings in Example 2.1.13 naturally arise in this context.

Lemma 2.1.10.3 indicates that there is a partial order on positive roots, with $\sum_{x\in\Delta_s} n_x x \leq \sum_{x\in\Delta_s} m_x x$ if each $n_x \leq m_x$. With respect to this ordering the simple roots are the minimal elements. We call a root **maximal** if it is maximal with respect to this ordering.

Proposition 2.2.1. *Let Δ be a root system with a connected Dynkin diagram. Then there is a unique maximal root, $\hat{\alpha}$.*

We refer to Proposition 4.5.8 in [8] for a proof, which relies on using properties of the associated simple Lie algebra introduced in 2.4.

We define the **extended Dynkin diagram** of a root system, Δ, $\hat{\Gamma}(\Delta)$, by taking $\Gamma(\Delta)$, adding a vertex corresponding to $-\hat{\alpha}$, and adding edges the same way as before: drawing $n_{-\hat{\alpha},x} \cdot n_{x,-\hat{\alpha}}$ edges from $-\hat{\alpha}$ to $x \in \Delta_s$, adding an arrow if the roots have different length. Writing $\hat{\alpha} = \sum_x \kappa_x x$, we have a distinguished labelling of $\hat{\Gamma}(\Delta)$, called the **primitive isotropic labelling**, $\kappa \in \mathbb{R}^{V(\hat{\Gamma}(\Delta))}$, which has $(-\hat{\alpha})$-entry equal to 1 and x-entry equal to κ_x for $x \in \Delta_s$. These diagrams turn out to agree with the ones in Example 2.1.13 and the primitive isotropic labellings agree with the labellings in the Example as well.

The reason we prefer $-\hat{\alpha}$ over $\hat{\alpha}$ is that $n_{\hat{\alpha},x},\ n_{x,\hat{\alpha}} > 0$ (see Proposition 4.59 of [8]). But the Cartan integers of distinct simple roots were negative and so choosing $-\hat{\alpha}$, which has $n_{-\hat{\alpha},x},\ n_{x,-\hat{\alpha}} < 0$, is more natural when considering the Cartan matrix corresponding to $\hat{\Gamma}(\Delta)$.

Example 2.2.2. In the case of A_n, where $\Delta^+ = \{e_i - e_j : i < j\}$, we find that the maximal root is $\hat{\alpha} = e_1 - e_{n+1} = \sum_{i=1}^n e_i - e_{i+1}$. This root connects to $e_1 - e_2$ and $e_n - e_{n+1}$ by single edges and so we get $\hat{A}_n$ with all vertices labelled with a 1. |

It is not by chance that the primitive isotropic labelling, $\kappa \in \mathbb{R}^{V(\hat{\Gamma}(\Delta))}$, is in the kernel of the Cartan matrix, C, derived from $\hat{\Gamma}(\Delta)$. The $(-\hat{\alpha})$-entry of $C\kappa$ is

$$2 + \sum_{x \in \Delta_s} v_x n_{-\hat{\alpha},x} = 2 + \frac{2}{\langle \hat{\alpha}, \hat{\alpha} \rangle} \sum_{x \in \Delta_s} \kappa_x \langle -\hat{\alpha}, x \rangle = 2 - 2\frac{\langle \hat{\alpha}, \hat{\alpha} \rangle}{\langle \hat{\alpha}, \hat{\alpha} \rangle} = 0.$$

Similarly, for $x \in \Delta_s$, the x-entry of $C\kappa$ is

$$n_{x,-\hat{\alpha}} + \sum_{y \in \Delta_s} \kappa_y n_{x,y} = \frac{2\langle x, -\hat{\alpha} \rangle}{\langle x, x \rangle} + \frac{2}{\langle x, x \rangle} \sum_{y \in \Delta_s} \langle x, \kappa_y y \rangle = \frac{2\langle x, -\hat{\alpha} \rangle}{\langle x, x \rangle} + \frac{2\langle x, \hat{\alpha} \rangle}{\langle x, x \rangle} = 0$$

and so $C\kappa = 0$.

2.3. The Weyl group

In this section, we introduce the Weyl group and describe how it helps recover the root system of a Dynkin diagram from the set of simple roots. As a result, there is a one-to-one correspondence between Dynkin diagrams and root systems. In the later chapters, the Weyl group will be crucial in our proof of Gabriel's theorem.

Definition 2.3.1. *Given a root system, $\Delta \subseteq V$, the corresponding **Weyl group**, $W = W(\Delta)$, is a subgroup of $GL(V)$ generated by the reflections σ_α, for $\alpha \in \Delta$.*

Notice that by property 2.1.1.d, W sends roots to roots and hence is a subgroup of the symmetric group S_Δ. So W is finite. We have the additional useful properties

Proposition 2.3.2. *If W is the Weyl group of Δ, with Δ_s a set of simple roots, then:*

1. *W is generated by the **simple reflections**, σ_x for $x \in \Delta_s$*
2. *$\Delta = \bigcup_{x \in \Delta_s} W \cdot x$*

Proof. We follow the proofs from [4]. We begin by proving a stronger version of (2). We claim that if $\alpha \in \Delta$, then there is a simple root $x \in \Delta_s$ and a series of simple reflections carrying x to α.

Without loss of generality, we may take α to be positive, since otherwise we may take a series of reflections carrying x to $-\alpha$ and then pre-compose with σ_x to carry x to α. Writing $\alpha = \sum_{y \in \Delta_s} n_y y$, we induct on the **height**, $\sum_{y \in \Delta_s} n_y \in \mathbb{N}$. If the height is 1, then α is simple and we are done.

Otherwise, the height is at least 2 and by 2.1.1.c, at least two of the $n_y > 0$. We can pick a simple root z so that $\langle z, \alpha \rangle > 0$. Such a z exists, since otherwise $\langle y, \alpha \rangle \leq 0$ for all simple roots, y, and

$$|\alpha|^2 = \sum_{y \in \Delta_s} n_y \langle y, \alpha \rangle \leq 0,$$

a contradiction. Observe that applying σ_z to α for this choice of z has no effect on the coefficients n_y, $y \neq z$, but changes n_z to becomes $n_z - \frac{2\langle z, \alpha \rangle}{\langle z, z \rangle} < n_z$. The root is still positive since the height being ≥ 2 implies that $n_y > 0$ for some $y \neq z$. Hence, $\sigma_z \alpha$ has a lower height than α, completing the inductive argument. This in fact shows that we can move from x to $\alpha \in \Delta^+$ through a sequence of positive roots.

To prove (1), we need to show that σ_α for $\alpha \in \Delta$, is a product of simple reflections. By the previous argument, we can write $\alpha = w \cdot x$ for $x \in \Delta_s$ and w a product of simple reflections. Then $\sigma_\alpha = \sigma_{w \cdot x} = w \cdot \sigma_x \cdot w^{-1}$, a product of simple reflections. To see that $\sigma_{w \cdot x} = w \cdot \sigma_x \cdot w^{-1}$, one may check that transformations agree on $w \cdot x$ and vectors orthogonal to $w \cdot x$. □

As a result, given a Dynkin diagram, Γ, with vertex set, V, we may recover the root system from the simple roots, i.e. describe which labellings of vertices of Γ, $v \in \mathbb{R}^V$, correspond to roots by considering the formal sum $\sum_{x \in V} v_x x$.

To do so, we start with a standard basis vector, $\epsilon_x \in \mathbb{R}^V$ and repeatedly apply various σ_y for $y \in V$, recording the vectors we find, until we generate all roots. Once our set of roots is closed under applying the σ_y, we pick a new ϵ_x and repeat. Given a $v \in \mathbb{R}^V$, the reflection σ_y is explicitly defined as

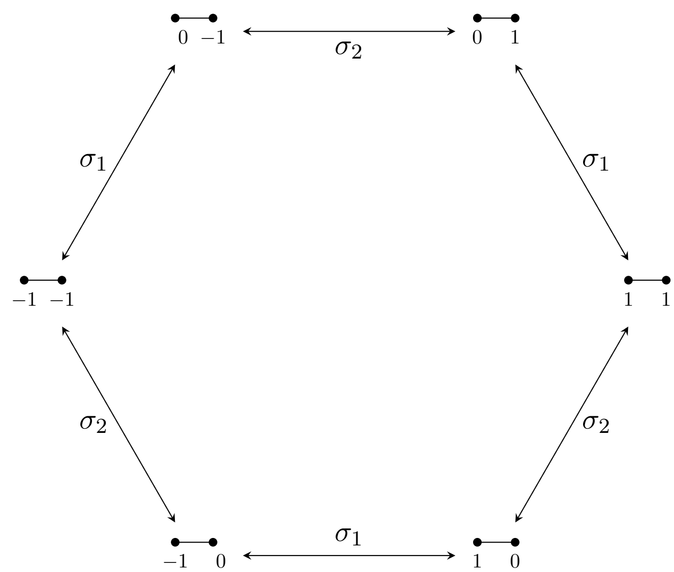
$$(\sigma_y(v))(x) = \begin{cases} v_x, & y \neq x \\ v_y - \sum_z v_z n_{y,z}, & y = x \end{cases} \tag{2.2}$$

where the $n_{y,z}$ can be recovered from the Dynkin diagram. In the case of simply laced Dynkin diagrams, this reduces to

$$(\sigma_y(v))(x) = \begin{cases} v_x, & y \neq x \\ -v_y + \sum_{z \sim y} v_z, & y = x \end{cases}. \tag{2.3}$$

Example 2.3.3. In the case of A_n, the Weyl group $W \cong S_{n+1}$, by identifying the simple reflections, $\sigma_{e_i - e_{i+1}}$, with the elementary transpositions, $s_i = (i\ i{+}1)$. Indeed, both systems of generators consist of order two elements and satisfy the braid relations, $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ and s_i, s_j commute when $|i - j| > 1$.

In the case special case of A_2, we show the resulting group action on the roots. Call $\sigma_{e_1-e_2} = \sigma_1$ and $\sigma_{e_2-e_3} = \sigma_2$. Going counterclockwise starting with the top left labelling, these correspond to the roots $-(e_2 - e_3),\ e_2 - e_3,\ e_1 - e_3,\ e_1 - e_2,\ -(e_1 - e_2),\ -(e_1 - e_3)$.



2.4. Application to semisimple Lie algebras

Root systems arise in the study of complex semisimple Lie algebras and help classify them. We will provide a brief exposition here without proving any of the details. We follow Chapters 1 through 4 of [8].

Recall that a complex Lie algebra, $\mathfrak{g}$, is **semisimple**, if it has no non-trivial solvable ideals or equivalently if it is a direct sum of simple Lie algebras (Lie algebras with no non-trivial ideals).

Proposition 2.4.1. *Let $\mathfrak{g}$ be a complex semisimple Lie algebra. Then it contains a nilpotent subalgebra, called the **Cartan subalgebra**, $\mathfrak{h}$, such that the normalizer $N_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}$. Furthermore, it is unique up to inner automorphism.*

Now we are able to define roots in the context of Lie algebras.

Definition 2.4.2. *Given* $\mathfrak{g}$ *with Cartan subalgebra* $\mathfrak{h}$ *and a dual vector* $\alpha \in \mathfrak{h}^*$*, we call*

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} : [H, X] = \alpha(H)X,\ \forall H \in \mathfrak{h}\}$$

the ***root space*** *of* α*, and*

$$\Delta = \{\alpha \in \mathfrak{h}^* \backslash 0 : \mathfrak{g}_\alpha \neq 0\}$$

are called the ***roots*** *of* $\mathfrak{g}$ *with respect to* $\mathfrak{h}$*.*

In a sense, the roots serve as generalized eigenvalues of $\mathfrak{h}$ acting on $\mathfrak{g}$ by the bracket, with the root space resembling an eigenspace. There is a decomposition of $\mathfrak{g}$ in the sense of vector spaces akin to diagonalization:

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$$

At present it is not clear, how Δ forms a root system as a subset of a $\mathbb{C}$-space $\mathfrak{h}^*$ with no given inner product. Without going into too much detail, there is a natural inner product on $\mathfrak{g}$ called the **Killing form**, given by $B(X, Y) = \mathrm{tr}(\mathrm{ad}_X \circ \mathrm{ad}_Y)$, where $\mathrm{ad}_X = [X, -]$. One can take a certain real form of $\mathfrak{h}$, $\mathfrak{h}_\mathbb{R}$, with $\mathfrak{h}_\mathbb{R}^*$ inheriting B as a real inner product. It then turns out that $\Delta \subseteq \mathfrak{h}_\mathbb{R}^*$ is indeed a root system.

Thus, we can define a Dynkin diagram corresponding to $\mathfrak{g}$, $\Gamma(\mathfrak{g})$.

Proposition 2.4.3. *Let* $\mathfrak{g}$ *be a semisimple Lie algebra with Cartan subalgebra* $\mathfrak{h}$*. Then:*

1. *The Dynkin diagram* $\Gamma(\mathfrak{g})$ *is independent of choice of Cartan subalgebra,* $\mathfrak{h}$
2. *Two semisimple Lie algebras,* $\mathfrak{g}, \mathfrak{g}'$ *are isomorphic, if and only if their diagrams* $\Gamma(\mathfrak{g}), \Gamma(\mathfrak{g}')$ *agree*
3. *Every Dynkin diagram in Theorem 2.1.8 is the Dynkin diagram of some Lie algebra*
4. $\mathfrak{g}$ *is simple, if and only if* $\Gamma(\mathfrak{g})$ *is connected*

5. *The simple summands of* $\mathfrak{g} = \bigoplus_i \mathfrak{g}_i$ *correspond to the connected components of* $\Gamma(\mathfrak{g})$

6. *The outer automorphisms of* $\mathfrak{g}$ *are in one-to-one correspondence with automorphisms of* $\Gamma(\mathfrak{g})$

We now list the Lie algebras corresponding to the Dynkin diagrams from the previous section

- A_n: $\mathfrak{sl}(n+1, \mathbb{C}) = \{A \in M_{n+1}(\mathbb{C}) : \mathrm{tr}(A) = 0\}$
- B_n: $\mathfrak{so}(2n+1, \mathbb{C}) = \{A \in M_{2n+1}(\mathbb{C}) : A + A^T = 0\}$
- C_n: $\mathfrak{sp}(n, \mathbb{C}) = \{A \in M_{2n}(\mathbb{C}) : AJ + JA^T = 0\}$, where $J = \left(\begin{smallmatrix} 0 & -I_n \\ I_n & 0 \end{smallmatrix}\right)$
- D_n: $\mathfrak{so}(2n, \mathbb{C}) = \{A \in M_{2n}(\mathbb{C}) : A + A^T = 0\}$.

The exceptional Lie algebras are again harder to describe. In the case of G_2, we may take the automorphism group of the Cayley numbers and consider the Lie algebra corresponding to this Lie group, which will have root system, G_2 [8].

We briefly mention the representation theory of semisimple Lie algebras as presented in [8]. A **representation** of semisimple complex Lie algebra, $\mathfrak{g}$, is a $\mathbb{C}$-linear map $\pi : \mathfrak{g} \to GL(V)$ for finite-dimensional V, such that $\pi([g, h]) = \pi(g)\pi(h) - \pi(h)\pi(g)$. Much like root spaces, are the "eigenvectors" of Cartan subalgebra $\mathfrak{h}$ acting on $\mathfrak{g}$, the **weight space** of $\mu \in \mathfrak{h}^*$, is given by $\mathfrak{h}$ acting on $\mathbb{C}^k$, via π:

$$V_\mu = \{x \in V : \pi(H)x = \mu(H)x, \ \forall H \in \mathfrak{h}\}$$

and the **weights** of π are the μ for which V_μ is non-zero. Let W_π be the set of weights. One gets the following decomposition analogous to the decomposition of $\mathfrak{g}$:

$$V = \bigoplus_{\mu \in W_\pi} V_\mu,$$

note that unlike the roots, weights can be zero, and the weight space, V_μ, can have dimension bigger than 1.

As it turns out, all the weights can be viewed as elements of real form $\mathfrak{h}_{\mathbb{R}}^*$. Then $W = \{\mu \in \mathfrak{h}_{\mathbb{R}}^* : \mu(X_\alpha) \in \mathbb{Z},\ \alpha \in \Delta\}$ is called the **weight lattice**, where X_α is a fixed representative of $\mathfrak{g}_\alpha$ We have the following theorem classifying irreducible representations of $\mathfrak{g}$. We have a partial order given on W, by taking $\mu_1 \leq \mu_2$ if $\mu_1(X_\alpha) \leq \mu_2(X_\alpha)$ for all $\alpha \in \Delta$. Then the following theorem classifies irreducible representations of π.

Theorem 2.4.4. *Let $\mathfrak{g}$ be a semisimple Lie algebra and let π be an irreducible representation. Then:*

1. *The weights $\mu \in W_\pi$ are elements of W.*
2. *There is a unique maximum element with respect to the partial order, called the **highest weight***
3. *Two irreducible representations are isomorphic, if and only if, they have the same highest weight*
4. *$\mu \in W$ is the highest weight of some irreducible representation, if and only if, it is a **dominant weight**, that is an element of W with the property $\mu(X_\alpha) \geq 0$ for all $\alpha \in \Delta^+$.*
5. *The dominant weights, $\mu \in W$, are sums of simple roots Δ_s.*

As an example, if $\mathfrak{g}$ is simple, then $\pi : \mathfrak{g} \to GL(\mathfrak{g})$, given by $\pi(g)h = [g, h]$ is an irreducible representation. The weights are just the roots and zero, $W_\pi = \Delta \cup \{0\}$. The maximal root $\hat{\alpha}$, defined in Proposition 2.2.1, is the highest weight of this representation.

We saw in this section, that any root system Δ, can be realized as the roots of some semisimple $\mathfrak{g}$. We also saw how certain sums of simple roots, the dominant weights, classify all the irreducible representations of $\mathfrak{g}$. We will in the next chapter, how quiver representations are classified by the positive roots.

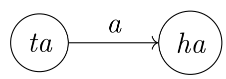
CHAPTER 3

Quiver representations

3.1. Basic terminology

We follow the definitions and notation of [2].

Definition 3.1.1. *A* ***quiver***, $Q = (V, E, h, t)$ *is a directed multi-graph with loops allowed.* V *denotes the set of vertices,* E *denotes the set of edges (also called arrows), and* $h, t : E \to V$ *return the* ***head*** *and* ***tail*** *of an arrow,* a*, depicted in the diagram below*



Arrows will be denoted by a, and vertices by x, y. The central objects of interest will be quiver representations. We denote by $\Gamma(Q)$, the underlying graph of Q: the undirected multi-graph obtained by forgetting arrow orientation.

Definition 3.1.2. *A* ***representation*** *of a quiver* $Q = (V, E, h, t)$*,* π*, is a map that attaches a* $\mathbb{C}$*-space,* $\pi(x)$ *to each vertex* $x \in V$*, and a* $\mathbb{C}$*-linear map,* $\pi(a) : \pi(ta) \to \pi(ha)$ *to each arrow* $a \in E$*.*

The ***dimension vector****,* $\underline{dim}(\pi) \in \mathbb{N}^V$*, of a representation, is a labelling of vertices attaching* $dim_{\mathbb{C}}(\pi(x))$ *to each* $x \in V$*.*

A **morphism**, ϕ, of quiver representations, π, ρ, of Q, attaches a $\mathbb{C}$-linear map $\phi(x) : \pi(x) \to \rho(x)$ for each $x \in V$, such that $\phi(ha)\pi(a) = \rho(a)\phi(ta)$ for all $a \in E$. With these morphisms, quiver representations of a fixed Q form a category, $\text{Rep}(Q)$. An isomorphism in this category can be viewed as a *change of basis* at each vertex, when the spaces $\pi(x) = \rho(x)$, agree.

It turns out this category is equivalent to the category of modules over a certain algebra, called the path algebra, $\mathbb{C}Q$. Thus, quiver representations are in one-to-one correspondence with algebra representations of a certain algebra, motivating their name.

Definition 3.1.3. *The* ***path algebra*** *of* Q, $\mathbb{C}Q$, *is a* $\mathbb{C}$*-algebra generated by directed paths in* Q, p, *where we allow for empty paths at a vertex, with empty paths denoted* e_x. *The product of two paths,* p, q, *is given by concatenation* pq *if the head of* q *agrees with the tail of* p, *and is zero otherwise. The algebra is associative and unital with* $1 = \sum_{x \in V} e_x$.

To see the equivilance of categories, take a $\mathbb{C}Q$-module, M, and define corresponding quiver representation, π, by letting $\pi(x) = e_x \cdot M$ and $\pi(a) = L_a|_{e_x \cdot M}$, where L_a is left multiplication by a, viewed as an element of $\mathbb{C}Q$ (a is a path consisting of a single arrow). To reverse the procedure, we define $M = \bigoplus_x \pi(x)$. The action of a is induced by $\pi(a)$ viewed as a map between appropriate direct summands of M. The action can be extended to longer paths, $p = a_k \cdots a_1$, by composing $\pi(a_k) \circ \cdots \circ \pi(a_1)$.

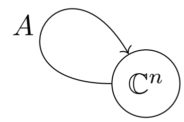
Remark 3.1.4. Notice that a path algebra, $\mathbb{C}Q$, is finite dimensional, if and only if there is a finite number of paths in Q, which is equivalent to Q having no directed cycles. |

Just like in representation theory, one may consider explicit matrix representations instead of abstract ones. We may do the same for Q. Within the category, Rep(Q), every representation is isomorphic to some matrix representation and so we may as well restrict our attention to these.

Definition 3.1.5. *A* ***matrix representation*** *of* Q, *of dimension* $\alpha \in \mathbb{N}^V$, *is an element* $\pi \in Rep_\alpha(Q) = \prod_{a \in E} M_{\alpha(ha), \alpha(ta)}(\mathbb{C})$, *where* $M_{k,l}(\mathbb{C})$ *denotes the set of* k*-by-*l *matrices over* $\mathbb{C}$. *This is viewed as a representation of* Q, *with* $\pi(x) = \mathbb{C}^{\alpha(x)}$

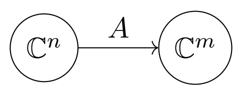
Example 3.1.6 (Jordan quiver)**.** Consider the quiver, L, consisting of a single vertex and a single arrow. We will call $\Gamma(L) =: \hat{A}_0$, since it shares many similarities with the extended Dynkin diagram family, $\hat{A}_n$, but does not fit in well with our earlier discussions (it is not simply laced and is not the extended Dynkin diagram of a root system).

A matrix representation of dimension n, is just some n-by-n matrix, A, attached to the arrow. An isomorphism of the matrix representations corresponds to a change of basis at the vertex, and so the isomorphism classes of representations of L are in one-to-one correspondence with Jordan normal forms.



The path algebra in this example is, $\mathbb{C}L \cong \mathbb{C}[a]$, isomorphic to the algebra of polynomials in indeterminant a. |

Example 3.1.7. Consider the following quiver, R, consisting of two vertices connected by a single arrow. Now isomorphism allows us to do a change of a basis at the domain and codomain, so the isomorphism classes of representations of R are in one-to-one correspondence with Smith normal forms. Notice that there are only finitely many possible m-by-n Smith normal forms, unlike in the previous example, where there are infinite n-by-n Jordan normal forms.



The path algebra in this example is, $\mathbb{C}R \cong \{\left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \in M_2(\mathbb{C})\}$, isomorphic to the algebra of upper triangular 2-by-2 matrices. |

The isomorphism classes in the set of matrix representations $\text{Rep}_\alpha(Q)$ are captured nicely by the following group action.

Definition 3.1.8. *The group* $GL_\alpha := \prod_{x \in V} GL_{\alpha(x)}(\mathbb{C})$ *acts on* $Rep_\alpha(Q)$*. If* $\phi \in GL_\alpha$ *and* $\pi \in GL_{\alpha(x)}(\mathbb{C})$*, then*

$$(\phi \cdot \pi)(a) = \phi(ha) \circ \pi(a) \circ \phi(ta)^{-1},$$

i.e. the matrices $\phi(x) \in GL_{\alpha(x)}(\mathbb{C})$ *describe a change of basis at* $\mathbb{C}^{\alpha(x)}$*. There is a natural correspondence between isomorphism classes of* α*-dimensional representations of* Q *and* GL_α*-orbits on* $Rep_\alpha(Q)$*.*

3.2. Simple and indecomposable representations

From the equivilance with the category of $\mathbb{C}Q$-modules, the notions of subrepresentation, direct sum, simple representations, and indecomposable representations carry over to the category of quiver representations.

Definition 3.2.1. *Let π, ρ, ψ be representations of Q. Then*

- *ρ is a **subrepresenation** of π, denoted $\rho \leq \pi$, if $\rho(x) \leq \pi(x)$ for all $x \in V$ and for all $a \in V$, $\rho(a) = \pi(a)|_{\rho(x)}$*
- *π is a **direct sum** of ρ, ψ, denoted $\pi = \rho \oplus \psi$, if $\pi(x) = \rho(x) \oplus \psi(x)$ for all $x \in V$ and $\pi(a) = \rho(a) \oplus \psi(a)$ for all $a \in E$*
- *π is **simple**, if it has no proper, non-trivial subrepresentations*
- *π is **indecomposable**, if it is not a direct sum of proper, non-trivial subrepresentations*

A simple representation is always indecomposable. However, unlike in the case of complex representation theory of groups and semisimple algebras, path algebras are often not semisimple and so indecomposable representations need not be simple. For brevity, we will write *indecomp iso classes* in place of *isomorphism classes of indecomposable representations.*

Example 3.2.2. Consider L from Example 3.1.6. A Jordan normal form breaks down into a direct sum of its Jordan blocks so an indecomposable representation can only consist of a single Jordan block. Hence, the indecomp iso classes correspond with k-by-k Jordan blocks $J_k(\lambda)$ for $\lambda \in \mathbb{C}$.

If $k > 1$, then the standard basis vector, e_1, is an eigenvector of $J_k(\lambda)$ and spans a subrepresentation with the restricted arrow map being multiplication by λ. Hence, the simple iso classes correspond to the 1-dimensional representations, $J_1(\lambda)$.

Consider R from Example 3.1.7 and matrix representation $A \in M_{m,n}(\mathbb{C})$. Then we decompose into three subrepresentations: a map from $\ker(A)$ to 0, a map from 0 to $\operatorname{im}(A)^{\perp}$, and an invertible map from $\ker(A)^{\perp}$ to $\operatorname{im}(A)$. The first two representations break down further into a direct sum of representations $\mathbb{C} \xrightarrow{0} 0$ and $0 \xrightarrow{0} \mathbb{C}$, respectively. After a change of basis of the domain and codomain, the invertible map becomes the identity and so breaks down into a sum of representations, $\mathbb{C} \xrightarrow{\text{id}} \mathbb{C}$.

Unlike L, which has infinite indecomp iso classes and simple iso classes, R only has 3 indecomp iso classes

$$\mathbb{C} \xrightarrow{0} 0 \qquad 0 \xrightarrow{0} \mathbb{C} \qquad \mathbb{C} \xrightarrow{\text{id}} \mathbb{C}$$

Out of the three of them only the first two are simple, since $0 \xrightarrow{0} \mathbb{C}$ is a subrepresentation of $\mathbb{C} \xrightarrow{\text{id}} \mathbb{C}$ (but $\mathbb{C} \xrightarrow{\text{id}} \mathbb{C}$ is still indecomposable since it is not a direct sum of $0 \xrightarrow{0} \mathbb{C}$ and $\mathbb{C} \xrightarrow{0} 0$). |

Example 3.2.2 leads to the following question: *When does a quiver have a finite number of simple iso classes (resp. indecomp iso classes)?* In the case of simple iso classes, the answer is quite straightforward.

Proposition 3.2.3. *A quiver, Q, has finite simple iso classes, if and only if it has no directed cycles. Furthermore, if Q has no directed cycles, then the simple iso classes are given by representations S_x for $x \in V$, where S_x attaches $\mathbb{C}$ to x, 0 to all other vertices, and the zero map to all edges.*

Proof. If Q has a directed cycle, p, at $x \in V$. Consider the representation, π_λ, for $\lambda \in \mathbb{C}^\times$, which attaches a copy of $\mathbb{C}$ to the vertices of p and attaches $\lambda \cdot \text{id}$ to each edge of p, with all other spaces and maps being set to zero. One can easily check that the representation is simple and that the π_λ are not isomorphic for distinct $\lambda \in \mathbb{C}^\times$.

Now suppose that Q has no directed cycle. Clearly, the S_x are simple, pairwise non-isomorphic representations. If π is simple, we may without loss of generality restrict to the induced subquiver, Q', on the supported vertices, $V' = \{x \in V : \pi(x) \neq 0\}$. Since Q' still has no directed cycle, there is a *sink* x, i.e. a vertex with no outgoing edges. Then $S_x \leq \pi$ forcing $\pi = S_x$, since π is simple. Thus, there are finitely many simple iso classes. □

The case of indecomposable representations is more complicated and beautiful and will be the focus of the following chapter. One important property of indecomposable representations is demonstrated by a theorem of Krull-Remak-Schmidt presented as Theorem 1.7.4 in [2], which states that the indecomp iso classes are the building blocks of all quiver representations.

Theorem 3.2.4 (Krull-Remak-Schmidt)**.** *Every quiver representation π is a finite direct sum of indecomposable representations, $\pi = \bigoplus_i \pi_i$, unique up to isomorphism and re-ordering of the π_i.*

CHAPTER 4

Gabriel's theorem

4.1. The statement

We call a quiver, Q, with finitely many indecomp iso classes, **finite type**. Let $\mathrm{Ind}(Q)$ denote the indecomp iso classes of Q.

Theorem 4.1.1 (Gabriel)**.** *Let Q be a connected quiver with underlying graph Γ. Then*

- *(a) Q has finite type, if and only if Γ is one of the A_n, D_n, E_n*
- *(b) Furthermore, if Γ is one of these simply laced Dynkin diagrams, then $\underline{dim} : Ind(Q) \to \mathbb{N}^V$ is a bijection between indecomp iso classes and the set of positive roots Δ^+.*

Here, we identify a dimesnion vector $v \in \mathbb{N}^V$ with a positive root $\alpha \in \Delta^+$, as $\alpha = \sum_{x \in V} v_x\, x$, since the vertices of Γ can be interpreted as simple roots of the associated root system. In particular, the second part of the theorem says that if there is an indecomposable representation with a certain dimension vector v, it is unique up to isomorphism. Notice also that the theorem does not depend on the orientation of the arrows of Q.

Example 4.1.2. Recall the case of A_2 in Example 2.3.3. The positive roots, $e_1 - e_2,\ e_2 - e_3,\ e_1 - e_3$, we encounter in the figure arise from the labellings, $\underset{1}{\bullet}\!\!-\!\!\underset{0}{\bullet}$, $\underset{0}{\bullet}\!\!-\!\!\underset{1}{\bullet}$, $\underset{1}{\bullet}\!\!-\!\!\underset{1}{\bullet}$. These are precisely the dimension vectors of the 3 indecomp iso classes seen in 3.2.2.

4.2. Tits form

To show the forward direction of Theorem 4.1.1.a, we introduce the following bilinear and quadratic forms.

Definition 4.2.1. *The **Euler form** of $Q = (V, E, h, t)$ is a bilinear form on $\mathbb{R}^V$,*

$$\langle \alpha, \beta \rangle_Q := \sum_{x \in V} \alpha(x)\beta(x) - \sum_{a \in E} \alpha(ta)\beta(ha)$$

*The **Tits form** of $Q = (V, E, h, t)$ is the corresponding quadratic form on $\mathbb{R}^V$,*

$$B_Q(\alpha) := \langle \alpha, \alpha \rangle_Q = \sum_{x \in V} \alpha(x)^2 - \sum_{a \in E} \alpha(ta)\alpha(ha).$$

*The **Cartan form**, is the symmetrized bilinear form $(\alpha, \beta)_Q = \langle \alpha, \beta \rangle_Q + \langle \beta, \alpha \rangle_Q$*

There is a motivation for this definition that comes from homological algebra. The two sums involved arise from the Hom and Ext functors, respectively. See Chapter 2 of Derksen and Weyman [2]. Notice that the Tits and Cartan form do not depend on orientation of edges, while the Euler form does.

The following is Lemma 4.1.3 from [2]

Lemma 4.2.2. *If Q has finite type, then B_Q is positive-definite.*

Proof. We begin by showing that $B_Q(\alpha) \geq 1$ for all non-zero $\alpha \in \mathbb{N}^V$. Recall, the group action defined in 3.1.8 and consider the dimension of the varieties GL_α, $\text{Rep}_\alpha(Q)$. We have $\dim(GL_\alpha) = \sum_{x \in V} \alpha(x)^2$ and $\dim(\text{Rep}_\alpha(Q)) = \sum_{a \in E} \alpha(ta)\alpha(ha)$. Hence, $B_Q(\alpha) = \dim(GL_\alpha) - \dim(\text{Rep}_\alpha(Q))$.

By Theorem 3.2.4, if Q has finite indecomp iso classes, then there are only finitely many ways to sum them to get an α-dimensional representation. Hence, there are finitely many α-dimensional iso classes and $\text{Rep}_\alpha(Q)$ has finitely many orbits under the GL_α action. Then one of these orbits must have full dimension in $\text{Rep}_\alpha(Q)$, say $GL_\alpha \cdot \pi$. Then by the orbit-stabilizer theorem for linear algebraic groups

$$B_Q(\alpha) = \dim(GL_\alpha) - \dim(\text{Rep}_\alpha(Q)) = \dim(GL_\alpha) - \dim(GL_\alpha \cdot \pi) = \dim((GL_\alpha)_\pi) \geq 1 \quad (4.1)$$

The stabilizer has to have dimension at least 1 since it contains the subgroup $\{\prod_{x\in V} \lambda \cdot \mathrm{id}_{\alpha(x)} : \lambda \in \mathbb{C}^\times\}$ which acts trivially on all elements of $\mathrm{Rep}_\alpha(Q)$.

Then for non-zero $\alpha \in \mathbb{Z}^V$, let $\alpha' = \mathbb{N}^V$ be the the vector with entries replaced by their absolute value. Then $B_Q(\alpha) \geq B_Q(\alpha') \geq 1$ by the definition of the Tits form. Considering non-zero $\alpha \in \mathbb{Q}^V$, one can scale by a large enough factor R so that $R\alpha \in \mathbb{Z}^V$ and so $B_Q(\alpha) \geq 1/R^2 > 0$. Finally, by continuity we get that B_Q is positive definite on $\mathbb{R}^V$. □

Observe, that if the underlying graph, Γ, is simply laced (has no multi-edges or loops), then $B_Q(\alpha) = \alpha^T(I - \frac{1}{2}A)\alpha$, where A is the adjaceny matrix of Γ. Thus, B_Q fails to be positive-definite if Γ has as an eigenvalue ≥ 2. Then $\hat{A}_n, \hat{D}_n, \hat{E}_n$, introduced in Example 2.1.13, cannot be the underling graphs of a finite-type Q, since their primitive isotropic labellings are 2-eigenvectors and thus make B_Q vanish. It follows by Remark 2.1.15, that B_Q is positive-semidefinite for these extended Dynkin diagrams, since their spectral radii are equal to 2.

We introduced $\hat{A}_0$ in Example 3.1.6. Similarly, we introduce the multi-graph $\hat{A}_1$ with two vertices connected by two edges, which was omitted earlier since it is not the extended Dynkin diagram of root system A_1. Then we may view the $\hat{A}_n$, $n \geq 0$, as cycle graphs on $n+1$ vertices. By labelling all vertices 1, the Tits form, B_Q, vanishes for these examples as well.

Lemma 4.2.3. *If B_Q is positive definite on Q, then it is also positive definite on any subquiver Q'.*

Proof. If $\alpha' \in \mathbb{R}^{V(Q')}$ is non-zero, then we may extend t to $\alpha \in \mathbb{R}^{V(Q)}$ by appending zeros. From the definition of the Tits form it easily follows that $B_{Q'}(\alpha') \geq B_Q(\alpha) > 0$ and so $B_{Q'}$ is positive-definite. □

Lemma 4.2.3 implies that if Q has finite type, $\Gamma(Q)$ cannot contain any of $\hat{A}_n, \hat{D}_n, \hat{E}_n$ as subgraphs. Since it doesn't contain $\hat{A}_0$, there are no loops and since it doesn't contain $\hat{A}_1$, there are no double edges. Thus, the graph $\Gamma(Q)$ is simply laced. Combined with Lemma 2.1.14, we see that $\Gamma(Q)$ must be one of the A_n, D_n, E_n, proving the forward direction of Theorem 4.1.1.a.

4.3. Reflection functors

Throughout this section the underlying graph of Q is one of the A_n, D_n, E_n. Proposition 2.3.2 tells us that the set of positive roots may be generated from the simple ones by applying the Weyl group action. In the same way, we hope to construct the indecomp iso classes corresponding to various positive roots from the simple representations, S_x, for $x \in V$, which correspond to the simple roots. Let $\epsilon_x := \underline{\dim}(S_x)$, the standard basis vectors. We thus wish to mimic the action of the Weyl group at the level of representations.

For $x \in V$, let $f_x(Q)$ denote the quiver with all arrows at x reversed. If a is an arrow, we will denote the reversed arrow by $\overline{a}$.

Definition 4.3.1. *Suppose Q has a sink at x, with incoming arrows $a_1, \ldots, a_k$. Then the **reflection functor** $C_x^+ : Rep(Q) \to Rep(f_x(Q))$ is defined as follows. If $\pi \in Rep(Q)$ and $\pi' = C_x^+(\pi)$, then let*

$$\phi : \bigoplus_i \pi(ta_i) \to \pi(x), \qquad \phi(v_1, \ldots, v_k) := \sum_i \pi(a_i)v_i$$

and define $\pi'(x) = ker(\phi) \subseteq \bigoplus_i \pi(ta_i)$ and $\pi'(\overline{a_i}) : ker(\phi) \to \pi(ta_i)$ to be the projection onto the i-th component of $\bigoplus_i \pi(ta_i)$. For the remaining vertices, $y \neq x$, and remaining arrows, $a \neq a_i$, define π' to agree with π.

Analogously, if Q has a source at x, with outgoing arrows, $a_1, \ldots, a_k$, we can define $C_x^- : Rep(Q) \to Rep(f_x(Q))$. We again define $\pi' = C_x^-(\pi)$ to agree with π away from x and the arrows a_i. Let

$$\psi : \pi(x) \to \bigoplus_i \pi(ha_i), \qquad \psi(v) = (\pi(a_1)v, \ldots, \pi(a_k)v)$$

and define $\pi'(x) = coker(\psi)$ and $\pi'(\overline{a_i}) : \pi(ha_i) \to coker(\psi)$ to be projection mod $im(\psi)$.

We will see that these two functors behave like the simple reflections, σ_x, of the Weyl group. The following is given as Theorem 4.3.9 in [2].

Proposition 4.3.2 (Bernstein-Gelfand-Ponomarev)**.** *Let x be a sink of Q, and π be an indecom-*

posable representation with $\underline{dim}(\pi) = \alpha$. Then

(a) *$\pi \cong S_x$, if and only if $C_x^+(\pi) = 0$. In this case, $\alpha = \epsilon_x$ and $\sigma_x(\alpha) = -\epsilon_x$.*

(b) *If $\pi \not\cong S_x$, then $C_x^+(\pi)$ is indecomposable with $\underline{dim}(C_x^+(\pi)) = \sigma_x(\alpha)$. Furthermore, $C_x^-(C_x^+(\pi)) \cong \pi$.*

(c) *The analogous statements hold with the roles of C_x^- and C_x^+ switched, if Q instead had a source at x.*

Proof. If $\pi \cong S_x$, then the kernel of the map ϕ is trivial and so $C_x^+(S_x) = 0$. Conversely, if $C_x^+(\pi) = 0$, then for $y \neq x$, $\pi(y) = 0$, which forces all the arrows to have zero maps. Then $\pi(x) = \mathbb{C}^n$ for some n. Since π is indecomposable, $n = 1$ and $\pi \cong S_x$.

Now suppose $\pi \not\cong S_x$, then we claim that ϕ is onto. Otherwise, $\pi(x) = \operatorname{im}\phi \oplus (\operatorname{im}\phi)^\perp$. Let ρ be the subrepresentation of ϕ, with $\pi(x)$ replaced by $\operatorname{im}(\phi)$ (we just restrict the codomain of the maps $\pi(a_i)$). Let τ be the subrepresentation, with $\tau(x) = (\operatorname{im}\phi)^\perp$ and all other $\tau(y) = 0$. Then $\pi = \tau \oplus \rho$. Since we supposed for contradiction that ϕ was not onto, $\tau \neq 0$ and we cannot have $\rho = 0$ since otherwise $\pi = \tau$ would be forced to be S_x. This contradicts the fact that π is indecomposable.

Thus, ϕ is onto and by rank-nullity theorem

$$\dim(C_x^+(\pi)(x)) = \dim(\ker\phi) = \dim\left(\bigoplus_i \pi(ta_i)\right) - \dim(\operatorname{im}\pi) = \sum_{y\sim x}\alpha(z) - \alpha(x) = \sigma_x(\alpha)(x)$$

where the last equality follows from (2.3). Since the spaces at the other vertices are unchanged, $\underline{\dim}(C_x^+(\pi)) = \sigma_x(\alpha)$.

Now we show $C_x^-(C_x^+(\pi)) = \pi$. Let $\pi' = C_x^-(C_x^+(\pi))$. It agrees at vertices and arrows away from x. At x, the space becomes

$$\pi'(x) = \operatorname{coker}\left(\ker\phi \overset{\psi}{\hookrightarrow} \bigoplus_i \pi(ta_i)\right) \cong \frac{\bigoplus_i \pi(ta_i)}{\ker\left(\bigoplus_i \pi(ta_i) \overset{\phi}{\to} \pi(x)\right)} \cong \operatorname{im}\phi = \pi(x).$$

For $v_i \in \pi(ta_i)$, where $\pi(ta_i)$ may be identified as a subspace of $\bigoplus_i \pi(ta_i)$,

$$\pi'(a_i)(v_i) = v_i + \ker\phi$$

which is identified by the first isomorphism theorem with $\phi(v_i) = \pi(a_i)v_i \in \operatorname{im}\phi = \pi(x)$. Hence $\pi'(a_i)$ is identified with $\pi(a_i)$ and $\pi' \cong \pi$.

Now suppose for contradiction that $C_x^+(\pi) = \rho_1 \oplus \cdots \oplus \rho_t$ is decomposable for ρ_i simple with $t > 1$. Note that none of the ρ_i may be isomorphic to S_x, since this would correspond to a non-zero vector $v \in \ker\phi$ with projections onto all $\pi(ta_i)$ components being zero. Then $\pi = C_x^-(C_x^+(\pi)) = \bigoplus_i C_x^-(\rho_i)$ a sum of non-zero subrepresentations since none of the ρ_i are S_x. This contradicts, the fact that π was indecomposable. Part (c) follows by similar reasoning. □

Unlike with the elementary reflections we must be careful when applying a series of the reflection functors, C_x^+, C_x^-, since they can only be applied when x is a sink (resp. source). This is because the σ_x, were able to send positive roots to negative ones. However, whenever applying $C_x^\pm$ would result in a representation with a negative dimension vector, it will instead send the representation to zero. Since we reverse arrows as we apply these functors, we are led to the following definition.

Definition 4.3.3. *A sequence of sinks (resp. sources), $s = (x_1, \ldots, x_k) \in V(Q)$, is* ***admissible*** *if x_1 is a sink (resp. source) of Q and for $i > 1$, x_i is a sink (resp. source) of $f_{x_{i-1}} \circ \cdots \circ f_{x_1}(Q)$.*

We have the following simple corollary of Proposition 4.3.2, given as Lemma 4.4.2 in [2].

Corollary 4.3.4. *Let $s = (x_1, \ldots, x_m)$ be an admissible sequence of sinks of Q and let $Q' = f_{x_m} \circ \cdots \circ f_{x_1}(Q)$. Let $C_s^+ : C_{x_m}^+ \cdots C_{x_1}^+ : Rep(Q) \to Rep(Q')$ and $C_s^- : C_{x_1}^- \cdots C_{x_m}^- : Rep(Q') \to Rep(Q)$. Then*

(a) $\mathcal{M} := \{C_{x_1}^- \cdots C_{x_{i-1}}^-(S_{x_i}) : i\}$ consists of indecomposable representations of Q
$\mathcal{M}' := \{C_{x_m}^+ \cdots C_{x_{i+1}}^+(S_{x_i}) : i\}$ consists of indecomposable representations of Q'

(b) For $\pi \in Ind(Q)$, $C_s^+(\pi)$ is zero if $\pi \in \mathcal{M}$. Otherwise, $C_s^+(\pi)$ is indecomposable and

$C_s^-(C_s^+(\pi)) = \pi$. The analogous statement holds for C_s^- and $\mathcal{M}'$.

(c) C_s^+ induces a bijection between Ind$(Q)\backslash\mathcal{M}$ and Ind$(Q')\backslash\mathcal{M}'$, with inverse is given by C_s^-.

Now recall that we are assuming Q is one of A_n, D_n, E_n and hence $\Gamma(Q)$ is a tree and cannot have directed cycles. Nevertheless, we have the following more general statement.

Proposition 4.3.5. *If Q has no directed cycles, then there exists and admissible sequence of sinks, $(x_1, \ldots, x_n)$, without repeats, such that $V(Q) = \{x_1, \ldots, x_n\}$*

Proof. We begin by **topologically sorting** $V(Q)$. That is we label the vertices $x \in V(Q)$, $x_1, \ldots, x_n$ such that whenever there is an arrow from x_i to x_j, $i > j$. We do this as follows. Since Q has no directed cycles, it has a sink; call it x_1. Then remove x_1 and all of its arrows from Q. This quiver is again acyclic and hence has a sink; call it x_2. Proceed this way until all the vertices have been given an index. If there was an arrow from x to y, then it is not possible for x to be a sink before y has been removed and so the index of y is lower than that of x.

We claim that $x_1, \ldots, x_n$ is an admissible sequence of sinks. We have that x_1 is a sink of Q. We need to show for all $i > 1$, x_i is a sink of $f_{x_{i-1}} \circ \cdots \circ f_{x_1}(Q)$. For all x_j, with $j > i$, an arrow between x_i, x_j, must be pointing from x_j to x_i in Q and is not flipped by any of the $f_{x_1}, \ldots, f_{x_{i-1}}$. Now consider x_j with $j < i$. An arrow between x_i, x_j, was pointing from x_i to x_j in Q, but got flipped by f_{x_j} and so is pointing away towards x_i in $f_{x_{i-1}} \circ \cdots \circ f_{x_1}(Q)$. Hence all arrows of $f_{x_{i-1}} \circ \cdots \circ f_{x_1}(Q)$ at x_i point inwards and x_i is a sink. □

Since each edge will be flipped exactly twice when applying $f_{x_n}, \ldots, f_{x_1}$, we have $Q = f_{x_n} \cdots f_{x_1}(Q)$. The corresponding functors $C^+ := C_{x_n}^+ \cdots C_{x_1}^+$, $C^- := C_{x_1}^- \cdots C_{x_n}^-$ from Rep(Q) to itself are called the **Coxeter functors**. They are independent of choice of admissible sink sequences up to natural equivilance (we refer to Lemma 4.4.6 of [2]).

4.4. Final steps

We are now ready to finish off the proof of Theorem 4.1.1. It is sufficient to prove part (b), since it implies the reverse direction of part (a). We need the following lemma

Lemma 4.4.1. *Let* $(x_1, \ldots, x_n)$ *be an admissible sequence of sinks for* Q *and define* $w = \sigma_{x_n} \ldots \sigma_{x_1}$, *which we will call the* ***Coxeter element*** *of the Weyl group,* W*. Then*

(a) If $\alpha \in \mathbb{R}^{V(Q)}$ *is fixed by* w*, then* $\alpha = 0$

(b) For non-zero $\alpha \in \mathbb{R}^{V(\mathbb{Q})}$*, there exists a* $k \geq 0$ *so that* $w^k(\alpha)$ *has a negative entry.*

Proof. Suppose non-zero α is fixed by w. As we apply the σ_{x_i}, the only time the value $\alpha(x_j)$ may change is when we apply σ_{x_j}. Thus, if α is fixed, then for all j

$$\alpha(x_j) = -\alpha(x_j) + \sum_{x_i \sim x_j} \alpha(x_i)$$

by (2.3) and so α is a 2-eigenvector of $\Gamma(Q)$. But this is a contradiction, since $\Gamma(Q)$ is one of the A_n, D_n, E_n.

For part (b), w has finite order since it is in the Weyl group of $\Gamma(Q)$, which is finite. Say the order is m. Then $\alpha + w\alpha + \cdots + w^{m-1}\alpha$ is fixed by w and hence must be zero. If α has a negative entry we are done and otherwise it must have a positive entry. Then $0 = \alpha + w\alpha + \cdots + w^{m-1}\alpha$ implies that one of the $w^k\alpha$ has a negative entry. □

Remark 4.4.2. The element w, may be defined for any ordering of vertices, and not just an admissible one. In any case, the order of w, called the *Coxeter number*, is independent of ordering. It turns out that for any root system, Δ, the order of Δ equals the rank of the root system times the Coxeter number of its Weyl group. See Corollary 3.32 of [5]. |

Let Ind(Q) denote the set of indecomp iso classes of Q. We complete the proof of Gabriel's theorem

Proposition 4.4.3. *The map* $\underline{dim} : Ind(Q) \to \Delta^+$ *is a bijection.*

Proof. First, we need to check that the dimension vector of any indecomposable representation, π, indeed corresponds to a positive root. Let $\alpha = \underline{\dim}(\pi)$ and let k be minimal such that $w^k(\alpha)$ has a negative entry, which exists by Lemma 4.4.1. Clearly, $k > 0$ since α has no negative entries. Then let j be minimal such that $\sigma_{x_j} \cdots \sigma_{x_1}(\sigma^{k-1}(\alpha))$ has a negative entry. Then $\sigma_{x_{j-1}} \cdots \sigma_{x_1}(\sigma^{k-1}(\alpha))$ has all positive entries and by Proposition 4.3.2 and Corollary 4.3.4, $\rho = C^+_{x_{j-1}} \cdots C^+_{x_1}(C^+)^{k-1}(\pi)$ is indecomposable, but, $C^+_{x_j}(\rho) = 0$. Hence $\rho \cong S_{x_j}$. Additionally, $\pi \cong (C^-)^{k-1}(C^-_{x_1} \cdots C^-_{x_{j-1}})(S_{x_j})$. Hence,

$$\alpha = \underline{\dim}(C^-)^{k-1}(C^-_{x_1} \cdots C^-_{x_{j-1}})(S_{x_j}) = w^{-(k-1)}\sigma_{x_1} \cdots \sigma_{x_{j-1}}(\epsilon_{x_j}) \in \Delta,$$

a positive root since α is a dimension vector.

If $\alpha \in \Delta^+$, then take k minimal and j minimal as before. Then since $\sigma_{x_j} \cdots \sigma_{x_1}(\sigma^{k-1}(\alpha))$ has a negative entry it is a negative root, but only differs from positive root $\sigma_{x_{j-1}} \cdots \sigma_{x_1}(\sigma^{k-1}(\alpha))$ at a single entry. Hence, all entries besides the x_j-entry are zero and $\sigma_{x_{j-1}} \cdots \sigma_{x_1} w^{k-1}(\alpha) = \epsilon_{x_j}$. Then defining $\pi = (C^-)^{k-1}(C^-_{x_1} \cdots C^-_{x_{j-1}})(S_{x_j}) \in \text{Rep}(Q)$, we have

$$\underline{\dim}(\pi) = w^{-(k-1)}\sigma_{x_1} \cdots \sigma_{x_{j-1}}(\epsilon_{x_j}) = \alpha$$

gives an indecomposable representation of dimension α.

Let ρ be another indecomposable α-dimensional representation, possibly not isomorphic to π. The sequence of simple reflections, σ_{x_i}, corresponding to the sequence of reflection functors in $C^+_{x_{j-1}} \cdots C^+_{x_1}(C^+)^{k-1}(\pi) \cong S_{x_j}$, never map a dimension vector to a negative root, starting with α. As a result, Corollary 4.3.4.b implies, that $C^+_{x_{j-1}} \cdots C^+_{x_1}(C^+)^{k-1}(\rho)$ is non-zero with dimension vector, ϵ_{x_j}. But this forces, $C^+_{x_{j-1}} \cdots C^+_{x_1}(C^+)^{k-1}(\rho) \cong S_{x_j}$ and $\rho \cong (C^-)^{k-1}(C^-_{x_1} \cdots C^-_{x_{j-1}})(S_{x_j}) = \pi$, proving injectivity of $\underline{\dim}$. □

CHAPTER 5

Applications and generalizations

5.1. Finite dimensional $\mathbb{C}$-algebras

There is a natural connection between quiver representations and algebra representations coming from the path algebra, $\mathbb{C}Q$, defined in 3.1.3. One consequence of Gabriel's Theorem 4.1.1 is that for finite type $Q = A_n, D_n, E_n$ with corresponding root system Δ, the indecomposable modules of $\mathbb{C}Q$ are counted by $|\Delta_+| = \frac{|\Delta|}{2}$, quantities listed in Theorem 2.1.8.

In this section, we will briefly described how to generalize beyond path algebras, to other $\mathbb{C}$-algebras. We follow Chapter 2 of [2].

Recall, that the *Jacobson radical* of a $\mathbb{C}$-algebra, A, is the two-sided ideal annihilating all simple A-modules

$$J(A) = \{w \in A : w \cdot M = 0,\ M \text{ simple}\} = \bigcap_{\mathfrak{m} \text{ maximal left ideal}} \mathfrak{m} = \bigcap_{\mathfrak{m} \text{ maximal right ideal}} \mathfrak{m}.$$

We claim that the maximal left ideals of $\mathbb{C}Q$ (which will turn out to be two-sided ideals) are $\mathfrak{m}_x := \langle a : a \in E(Q)\rangle + \langle e_y : y \in V(Q)\backslash\{x\}\rangle$. Indeed, they are maximal since $e_x \notin \mathfrak{m}_x$, but all other generators are and so $\mathbb{C}Q/\mathfrak{m}_x \cong \mathbb{C}e_x = \mathbb{C}$. Any other maximal ideal, $\mathfrak{m}$, may not contain all the e_x, since $1 = \sum_x e_x$. Thus, some $e_x \notin \mathfrak{m}$ and $\mathfrak{m} \subseteq \mathfrak{m}_x$. Then

$$J(\mathbb{C}Q) = \bigcap_{x \in V} \mathfrak{m}_x = \langle a : a \in E(Q)\rangle.$$

We will call an ideal of $\mathbb{C}Q$, I, **admissible** if $J(\mathbb{C}Q)^k \subseteq I \subseteq J(\mathbb{C}Q)^2$ for some $k \geq 2$. In other words, an ideal I is admissible if it has a generating set of elements w, where w are sums of paths of lengths between 2 and k. While $\mathbb{C}Q$ may be infinite dimensional in the case when there is a directed cycle by Remark 3.1.4, $\mathbb{C}Q/I$ is finite dimensional since it is a quotient of $\mathbb{C}Q/J(C\mathbb{Q})^k$, whose dimension is bounded by the number of paths of length at most k. Furthermore,

$$(\mathbb{C}Q/I)/(J(\mathbb{C}Q/I)) \cong \mathbb{C}Q/J(\mathbb{C}Q) \cong \mathbb{C}\langle e_x : x \in V(Q)\rangle \cong \mathbb{C}^{\oplus n},$$

where $n = |V(Q)|$ and $\mathbb{C}^{\oplus n}$ is the $\mathbb{C}$-algebra obtained by taking the n-fold direct sum of $\mathbb{C}$.

We will call a finite dimensional $\mathbb{C}$-algebra, A, **basic**, if $A/J(A) \cong \mathbb{C}^{\oplus n}$ for some n. We saw above that $\mathbb{C}Q/I$ is basic for admissible I. It turns out that all basic algebras are of this form

Theorem 5.1.1. *For any basic algebra, A, there is a unique quiver, Q, and some (possibly non-unique) admissible ideal $I \subseteq \mathbb{C}Q$, such that $A \cong \mathbb{C}Q/I$.*

We sketch the construction of Q for a given A and refer the reader to the proof of Theorem 3.3.2 in [2] for more details. The quotient, $A/J(A)$, has n orthogonal idempotents and it turns out that they may be lifted to orthogonal idempotents of A, $e_1, \ldots, e_n$. Then let Q consists of vertices $e_1, \ldots, e_n$ and draw $\dim_{\mathbb{C}}(e_j M e_i)$ arrows from vertex e_i to vertex e_j, where $M = J(A)/J(A)^2$.

An alternative construction seen in [1], has a vertex for every simple module isomorphism class of A, S_i, and adds $\dim_{\mathbb{C}} \mathrm{Ext}^1(S_i, S_j)$ arrows from S_i to S_j. For this reason, Q is sometimes called the Ext^1-quiver of A.

In general, a finite dimensional A, has $A/J(A)$ semisimple and, by Wedderburn-Artin theorem, is isomorphic to a direct sum of $M_n(\mathbb{C}), M_n(\mathbb{H})$, for various n. It turns out that despite seeming like a small class of algebras, basic algebras reach all finite dimensional algebras by Morita equivilance. Recall that two algebras A, B are **Morita equivalent** if the categories of A-modules and B-modules are equivalent.

Theorem 5.1.2. *Any finite dimensional $\mathbb{C}$-algebra, A, is Morita equivalent to some basic algebra B. More concretely, write $A = P_1^{\oplus m_1} \oplus \cdots \oplus P_n^{\oplus m_n}$ as a sum of pairwise non-isomorphic, indecomposable P_i. Then we may take B to be the opposite algebra of $End_A(P_1 \oplus \cdots \oplus P_n)$.*

*If A is also **hereditary**, that is, all submodules of projective A-modules are projective, then this*

$B \cong \mathbb{C}Q$ for some Q.

Since the number of indecomp iso classes of A-modules is an invariant of Morita equivilance, Gabriel's theorem extends to hereditary finite dimensional $\mathbb{C}$-algebras. To see if a hereditary A has finite type, one may use the above constructions to associate a quiver and check if it is one of the A_n, D_n, E_n.

5.2. Quivers of non-finite type

Gabriel's theorem tells us that for finite type quivers, the set of dimension vectors that are encountered among indecomposable representations is the set of positive roots. It is natural to ask what this set looks for non-finite type quivers. We follow chapters 6 and 7 of [5] where most of the proofs of these statements can be found. Throughout this section Q is a quiver with no loops ($\hat{A}_0$ is not a sub-quiver).

To the underlying graph, $\Gamma(Q)$, we associate a generalized Cartan matrix defined using the same procedure as in Remark 2.1.11. Then there is an associated Kac-Moody Lie algebra and define the **roots** of Q, $\Delta(Q)$, to be the roots of this Lie algebra, defined as in 2.4.2.

There is an equivalent but more direct definition of the roots corresponding to a quiver. Define the Weyl group, $W(Q)$, associated to an undirected multi-graph, $\Gamma(Q)$, by letting it be the group generated by reflections on $\mathbb{R}^{V(\Gamma)}$, given by a slight generalization of equation (2.3) that accounts for multi-edges,

$$(\sigma_y(\alpha))(x) = \begin{cases} \alpha(x), & y \neq x \\ -\alpha(y) + \sum_{z \in V} d_{z,y} \cdot \alpha(z), & y = x \end{cases}, \tag{5.1}$$

where $d_{z,y}$ is the number of edges between z, y. This is equivalent to $\sigma_y(\alpha) = \alpha - (\alpha, \epsilon_y)_Q \epsilon_y$, where $(,)_Q$ is the Cartan form defined in 4.2.1.

Definition 5.2.1. *The* ***roots*** *of Q, $\Delta \subseteq \mathbb{Z}^{V(Q)}$, are defined as follows. The* ***real roots*** *are $\Delta_{re} := \bigcup_{x \in V(Q)} W \cdot \epsilon_x$.*

Let

$$K := \{\alpha \in \mathbb{N}^{V(Q)} \backslash 0 : \ \alpha(x) \leq (\sigma_x(\alpha))(x) \ \forall x, \ supp(\alpha) \ connected \ in \ \Gamma(Q)\}.$$

Then the ***imaginary roots*** *are* $\Delta_{im} := W \cdot K \sqcup (-W \cdot K)$*. Then* $\Delta := \Delta_{re} \cup \Delta_{im}$*.*

It is not trivial, but can be shown that the real and imaginary roots both split into positive and negative roots based on if they have all positive or all negative entries. An easy computation shows that $B_Q(\sigma_x \alpha) = \alpha$ and so B_Q is invariant under the Weyl group. In particular, all $\alpha \in \Delta_{re}$ have $B_Q(\alpha) = 1$. But for the condition α being in K is equivalent to $(\alpha, \epsilon_x)_Q \leq 0$ and so

$$2B_Q(\alpha) = (\alpha, \alpha)_Q = \sum_{x \in V} \alpha(x)(\alpha, \epsilon_x)_Q \leq 0,$$

so the sets of real and imaginary roots are disjoint.

As we will see later, the generalized set of roots precisely agrees with the set, $\{\alpha : \mathrm{Ind}_\alpha(Q) \neq \varnothing\}$.

We say that $\pi \in \mathrm{Ind}(Q)$ is **preprojective** if $(C^+)^n(\pi) = 0$ for some n, **preinjective** if $(C^-)^n(\pi) = 0$ for some n, and **regular** otherwise. Note that a representation may be both preprojective and preinjective, which is the case for all indecomposable representations of a finite type Q. It is a fact, that $\pi \in \mathrm{Rep}(Q)$ is projective (resp. injective) when viewed as a $\mathbb{C}Q$-modules, if and only if $C^+(\pi) = 0$ (resp. $C^-(\pi) = 0$), motivating the names.

5.2.1. Extended Dynkin diagrams

Let Q, be one of A_n, D_n, E_n and $\hat{Q}$ be the corresponding extended Dynkin diagram, $\hat{A}_n, \hat{D}_n, \hat{E}_n$, with auxiliary vertex y_0. Recall, that κ is the primitive isotropic labelling in Example 2.1.13. By Remark 2.1.15, subgraphs of $\Gamma(\hat{Q})$ are of ADE-type.

Theorem 5.2.2. *Let* $\Gamma(\hat{Q})$ *be an extended Dynkin diagram. Then:*

1. $\Delta_{re} = \{\alpha \in \mathbb{Z}^V : B_{\hat{Q}}(\alpha) = 1\}$

2. $\Delta_{im} = (\mathbb{Z}\kappa) \backslash 0 = \{\alpha \neq 0 : B_{\hat{Q}}(\alpha) = 0\}$

3. The Weyl group is isomorphic to the internal semidirect product

$$W(\hat{Q}) \cong W(Q) \ltimes \langle \tau_\alpha : \alpha \in \mathbb{Z}^{V(Q)} \rangle,$$

where $\tau_\alpha(\beta) = \beta + (\alpha, \beta)_Q \kappa$ *for* $\beta \in \mathbb{R}^{V(Q)}$*. The group* $\langle \tau_\alpha : \alpha \in \mathbb{Z}^{V(Q)} \rangle \cong \mathbb{Z}^{|V(Q)|}$ *and so* $W(\hat{Q})$ *is infinite.*

Example 5.2.3. The Kronecker quiver, K, has $\Gamma(K) = \hat{A}_1$, with both edges pointed in the same direction.

$$x \rightrightarrows y$$

The real roots have the form $\Delta_{re} = \{(n, n+1)\} \cup \{(n+1, n)\}$ and the imaginary roots are $\Delta_{im} = \{(n, n) : n \neq 0\}$. We now list all the indecomp iso classes. The preprojective indecomp iso classes are given by

$$\mathbb{C}^n \underset{\begin{bmatrix} 0 \\ I_n \end{bmatrix}}{\overset{\begin{bmatrix} I_n \\ 0 \end{bmatrix}}{\rightrightarrows}} \mathbb{C}^{n+1}$$

and the preinjective iso classes are of the form

$$\mathbb{C}^{n+1} \underset{[0\ I_n]}{\overset{[I_n\ 0]}{\rightrightarrows}} \mathbb{C}^n$$

The regular indecomp iso classes are parameterized by $[\lambda : \mu] \in \mathbb{CP}^1$:

$$\mathbb{C}^n \underset{J_n(\mu)}{\overset{J_n(\lambda)}{\rightrightarrows}} \mathbb{C}^n$$

The Kronecker quiver ends up being crucial to the classification of indecomposable representations for general extended Dynkin diagrams. We refer the reader to Theorem 7.41 of [5].

Theorem 5.2.4. *For $\Gamma(\hat{Q})$ be an extended Dynkin diagram:*

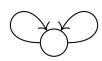
1. *$Ind_\alpha(\hat{Q}) \neq \varnothing$, if and only if $\alpha \in \Delta^+$*

2. *If $\alpha \in \Delta^+_{re}$, there is a unique indecomp iso class which is preprojective or preinjective*

3. *If $\alpha \in \Delta^+_{im}$, then there are infinite indecomp iso classes which are regular and $Ind_\alpha(Q)$ is parameterized in a natural way by $(\mathbb{CP}^1 \backslash D) \cup P$, where D, P are finite sets.*

5.2.2. Wild quivers

Consider a vertex with two loops



It has path algebra $k\langle x, y\rangle$, generated by non-commuting indeterminants, x, y. The problem of finding canonical forms for the indecomp iso classes, is equivalent to simultaneous block diagonalization of two matrices $A, B \in M_n(\mathbb{C})$, which is called the *wild problem* and is quite complicated. We say an algebra is wild if it contains the representation theory of this quiver. The statement is formalized as follows

Definition 5.2.5. *A $\mathbb{C}$-algebra, A, is **wild** if there is an A-$k\langle x, y\rangle$-bimodule, M, such that the functor $F : k\langle x, y\rangle$-mod $\to$ A-mod, $X \mapsto M \otimes_{k\langle x,y\rangle} X$ has $F(X)$ indecomposable, if and only if X indecomposable and $F(X) \cong F(Y)$, if and only if $X \cong Y$.*

*If A is not wild, then it is **tame**.*

Theorem 5.2.6 (Drozd)**.** *The path algebra of a quiver, Q, is wild, if and only if $\Gamma(Q)$ is not a Dynkin nor an extended Dynkin diagram.*

Furthermore, wild quivers are classified by having imaginary roots $\alpha \in \Delta_{im}$ such that $B_Q(\alpha) < 0$.

This is not the case for tame quivers, since B_Q is positive semi-definite when $\Gamma(Q)$ has spectral radius ≤ 2. We finish the section by stating a theorem of Kac that greatly generalizes Gabriel's theorem

Theorem 5.2.7 (Kac)**.** *If Q is a quiver without loops, then*

1. *$Ind_\alpha(Q) \neq \varnothing$, if and only if $\alpha \in \Delta^+$*
2. *If $\alpha \in \Delta^+_{re}$, there is a unique indecomp iso class*
3. *If $\alpha \in \Delta^+_{im}$, then there are infinite indecomp iso classes parameterized by a union of algebraic varieties $Z_1, \ldots, Z_N$, with maximal dimension $max(dim(Z_i)) = 1 - B_Q(\alpha)$*

Notice that because wild quivers have $\alpha \in \mathbb{N}^V$ with $B_Q(\alpha) < 0$, $\mathrm{Ind}_\alpha(Q)$ has parameterizing varieties of dimension ≥ 2. This gives an alternative characterization of tame quivers as the ones with each $\mathrm{Ind}_\alpha(Q)$ being parameterized by a finite set of points and curves. This gives further insight into why the wild problem is so difficult.

5.3. ADE-correspondence for finite subgroups of $SU(2)$

We finish the paper, with another example of ADE-correspondence: the finite subgroups $G \leq SU(2)$ are in one-to-one correspondence with the simply laced Dynkin diagrams. We follow Chapter 8 of [5].

Let $G \leq SU(2)$ be a finite non-trivial subgroup of $SU(2)$. Let $\{\rho_i\}$ be a set of representatives of the irreducible representations of G over $\mathbb{C}$. Let $\iota : G \hookrightarrow SU(2)$ be the representation given by inclusion (not necessarily irreducible). Now construct a multi-graph, $\Gamma(G)$, whose vertex set is $\{\rho_i\}$. For every pair of distinct vertices, ρ_i, ρ_j, draw $\dim_{\mathbb{C}}\mathrm{Hom}_G(\rho_i, \rho_j \otimes \iota)$ arrows between them, where $\mathrm{Hom}_G(A, B)$ is the space of G-invariant linear maps form A to B and $\rho_j \otimes \iota$ is the internal tensor product of G-representations.

We claim that $\dim_{\mathbb{C}}\mathrm{Hom}_G(\rho_i, \rho_j \otimes \iota)$ is symmetric in i, j, making Γ well-defined. Since $SU(2)$

consists of unitary matrices, for all $g \in G \subset SU(2)$, $\iota^*(g) = (g^{-1})^* = g = \iota(g)$. Then

$$\mathrm{Hom}_G(\rho_i, \rho_j \otimes \iota) \cong \mathrm{Hom}_G(\rho_i \otimes \iota^*, \rho_j) \cong \mathrm{Hom}_G(\rho_j, \rho_i \otimes \iota^*) \cong \mathrm{Hom}_G(\rho_j, \rho_i \otimes \iota),$$

where the second isomorphism follows by Schur's lemma. It turns out that the graphs, $\Gamma(G)$, obtained from various G are precisely the extended Dynkin diagrams, with the exception of $\hat{A}_0$. The following is known as the McKay correspondence

Theorem 5.3.1 (McKay)**.** *There is a one-to-one correspondence between finite subgroups, $G \leq SU(2)$ up to conjugation and extended Dynkin diagrams, associating the trivial subgroup with $\hat{A}_0$ and non-trivial G with $\Gamma(G)$, whose vertices correspond to the irreps of G.*

Furthermore, we may take the auxiliary vertex of the extended Dynkin diagram to be the trivial representation of G. The labelling of vertices assigning $\dim(\rho_i)$ to vertex ρ_i, is precisely the primitive isotropic labelling, κ.

We now present explicit constructions of these groups, first giving their presentations. Let non-trivial $G \leq SU(2)$ have $\Gamma(G) = \hat{\Gamma}_0$, where Γ_0 is one of the A_n, D_n, E_n. Recall by Remark 2.1.16, that a graph is a simply laced Dynkin diagram, if and only if it consists of three branches of lengths $c \geq b \geq a \geq 1$, satisfying $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} > 1$. Let a, b, c be the branch lengths associated to Γ_0. Then the group G given by the McKay correspondence has presentation

$$G \cong \langle x, y, z : x^a = y^b = z^c = xyz \rangle.$$

Note that for $\Gamma_0 = A_n$, $a = 1$ and there are multiple choices for b, c, as long as $b + c = n + 1$. For each of these choices, the presentation is isomorphic to the cyclic group, $\mathbb{Z}_{n+1}$, which is easily seen as a subgroup of $SU(2)$, by mapping $t \in \mathbb{Z}_{n+1}$ to $\begin{pmatrix} \omega^t & \\ & \omega^{-t} \end{pmatrix}$, for ω a primitive $(n+1)$-th root of unity.

For $\Gamma_0 = D_n$, $n \geq 4$, we have $a = b = 2$ and $c = n-2$. In this case, G is the **binary dihedral group**, BD_{4c}. It is obtained by taking the dihedral group of the c-gon and viewing it as a subgroup of $SO(3)$, by centering the c-gon at the origin. Then one uses the well-known double cover $SU(2) \to SO(3)$

to lift the dihedral group to an order $4c$ subgroup of $SU(2)$.

For $\Gamma_0 = E_6, E_7, E_8$, we have $a = 2$, $b = 3$ and $c = 3, 4, 5$. Observe that in each case, "$\{b, c\}$" is the Schlafli symbol of a regular polyhedron. Here, the Schlafli symbol, $\{b, c\}$, corresponds to a regular polyhedron with b-gons as faces and c faces at each vertex. Swapping b, c corresponds to taking the dual polyhedron: the cube for the octahedron, the dodecahedron for the icosahedron, and the tetrahedron being self-dual. Just as with the binary dihedral groups, we may take the regular polyhedron, X, corresponding to a Schlafli symbol, $\{3, 3\}, \{3, 4\}, \{3, 5\}$, and view its symmetries as a subgroup of $SO(3)$ by centering X at the origin. These are called the polyhedral groups, which are isomorphic for dual polyhedra. Upon lifting the polyhedral groups to $SU(2)$, we get the **binary polyhedral groups**.

We see how the double cover $SU(2) \to SO(3)$ also helps classify the subgroups of $SO(3)$, however, odd order cyclic groups $\mathbb{Z}_{2n+1}$ are not lifts of any subgroups of $SO(3)$. It will still turn out that the subgroups of $SO(3)$ are the cyclic groups of arbitrary order, dihedral groups, and polyhedral groups, which also have a natural ADE-correspondence. We summarize in the table below

Dynkin diagram	Group	(a, b, c)	$\lvert G\rvert$
A_n	$\mathbb{Z}_{n+1}$	$(1, b, n + 1 - b)$	n+1
D_n	$BD_{4(n-2)}$	$(2, 2, n - 2)$	$4(n - 2)$
E_6	binary tetrahedral	$(2, 3, 3)$	24
E_7	binary octahedral	$(2, 3, 4)$	48
E_8	binary icosahedral	$(2, 3, 5)$	120

Table 5.1: Finite subgroups of $SU(2)$

Notice that the order of the group is directly related to the triplet, a, b, c, via $\frac{4}{|G|} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} - 1$. See [5] for a simple proof of this formula.

BIBLIOGRAPHY

[1] CHAN, A. Quivers, their representation and applications, 2011. [Online:] https://aaronkychan.github.io/notes/pgsem/quivers.pdf.

[2] DERKSEN, H., AND WEYMAN, J. *An Introduction to Quiver Representations*. Graduate studies in mathematics. American Mathematical Society, 2017.

[3] GODSIL, C., AND ROYLE, G. F. *Algebraic Graph Theory*. No. Book 207 in Graduate Texts in Mathematics. Springer, 2001.

[4] IGUSA, K. *Lie Algebras*. 2011.

[5] KIRILLOV, A. *Quiver Representations and Quiver Varieties*. Graduate Studies in Mathematics. American Mathematical Society, 2016.

[6] MEHRLE, D. Root systems and dynkin diagrams.

[7] TANG, X. graph with eigenvalue at most 2. Math Stack Exchange, 2019. [Online:] https://math.stackexchange.com/questions/3169218/graph-with-largest-eigenvalue-at-most-2.

[8] ZILLER, W. *Lie Groups. Representation Theory and Symmetric Spaces*. 2021.