Monomial stability in Frobenius images

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S_n-representations and symmtric functions

A **representation** of S_n describes an action of S_n on some vector space V. Every S_n-representation uniquely *decomposes* into a direct sum of **irreducible components**, $\mathbb{S}^\lambda$, indexed by partitions $\lambda \vdash n$. Example for $n = 4$:

$$V \cong \mathbb{S}^{(2,1,1)} \oplus \mathbb{S}^{(2,1,1)} \oplus \mathbb{S}^{(4)} \oplus \mathbb{S}^{(3,1)}$$

The algebra of **symmetric functions**, Λ, contains formal power series in infinite variables, $F(x_1, x_2, \ldots)$, that are unaffected by permuting variables. The **Frobenius map** provides a bridge between S_n-representations and degree n symmetric functions

Frobenius image

$$\mathrm{Frob} : \text{“}S_n\text{-representations”} \longrightarrow \Lambda^{(n)}$$

$$\underset{\text{irreducibles}}{\mathbb{S}^\lambda} \longmapsto \underset{\text{Schur function}}{s_\lambda(x_1, x_2, \ldots)}$$

$$\underset{\text{indicator of conj class } C_\lambda}{1_{C_\lambda}} \longmapsto \underset{\text{scaled power sum funcion}}{\frac{|C_\lambda|}{n!} p_\lambda(x_1, x_2, \ldots)}$$

$$? \longmapsto \underset{\text{monomial basis}}{m_\lambda(x_1, x_2, \cdots)}$$

So the example above becomes

$$\mathrm{Frob}(V) = 2\, s_{(2,1,1)} + s_{(4)} + s_{(3,1)}$$

Representation stability

Church, Ellenburg, and Farb [2] introduced **representation stable** sequences

$$\overset{S_1}{\overset{\circlearrowright}{V_1}} \xrightarrow[\varphi_1]{} \overset{S_2}{\overset{\circlearrowright}{V_2}} \xrightarrow[\varphi_2]{} \overset{S_3}{\overset{\circlearrowright}{V_3}} \xrightarrow[\varphi_3]{} \cdots \tag{1}$$

where each V_n is an S_n-representation and the multiplicities of irreducibles *stabilize*.

Given $\lambda = (\lambda_1, \cdots, \lambda_l) \vdash k$, define

$$\lambda[n] := (n-k, \lambda_1, \lambda_2, \cdots, \lambda_l) \vdash n$$

for $n \geq k + \lambda_1$.

Definition

A sequence of the form (1) is **uniformally representation multiplicity stable (URMS)** if $\exists N$, s.t. for all λ and $n \geq N$, the multiplicity of $\mathbb{S}^{\lambda[n]}$ in V_n does not depend on n.

Representation stability in [2]= URMS + some extra conditions on the maps φ_n.

$$V_n = \mathbb{S}^{(n-2,1,1)} \oplus \mathbb{S}^{(n-2,1,1)} \oplus \mathbb{S}^{(n)} \oplus \mathbb{S}^{(n-1,1)}$$

is URMS with stable range $n \geq 2$.

This phenomena has been observed in many scenarios: cohomology of configuration spaces, pure braid groups, flag varieties, homology of certain Torelli subgroups [2].

Monomial and character stability

Given the Schur expansion

$$\mathrm{Frob}(V_n) = \sum_\lambda c_{\lambda,n} s_{\lambda[n]}, \tag{2}$$

define $\mathrm{rg}^s_\lambda(V_\bullet)$ to be the smallest N, such that the coefficients have stabilized

$$c_{\lambda,N} = c_{\lambda,N+1} = c_{\lambda,N+2} = \cdots.$$

$V_\bullet$ being URMS is equivalent to $\sup_\lambda \mathrm{rg}^s_\lambda(V_\bullet) < \infty$. One can similarly define $\mathrm{rg}^m_\mu(V_\bullet)$ to track when the coefficient $d_{\mu,n}$ stabilizes in

$$\mathrm{Frob}(V_n) = \sum_\mu d_{\mu,n} m_{\mu[n]}. \tag{3}$$

Proposition (B. 2025+)

Every Schur coefficient, $c_{\lambda,n}$, stabilizes if and only if every monomial coefficient, $d_{\mu,n}$, stabilizes.

Main Result (B. 2025+)

Define the **weight** of a sequence $V_\bullet$

$$\mathrm{wt}(V_\bullet) = \sup\{|\lambda| : c_{\lambda,n} \neq 0,\ \text{some } n\}.$$

(a) A sequence $(V_n)_n$ with V_n an S_n-rep is URMS if and only if $\mathrm{wt}(V) < \infty$ and the monomial coefficients $d_{\mu,n}$ in (3) stabilize. In this case, the multiplicity of $\mathbb{S}^{\lambda[n]}$ in V_n stabilizes once

$$n \geq \max_{|\mu| \leq |\lambda|} \mathrm{rg}^m_\mu(V_\bullet). \tag{4}$$

A uniform stable range of $V_\bullet$ is given by $n \geq \max_{|\mu| \leq \mathrm{wt}(V_\bullet)} \mathrm{rg}^m_\mu(V_\bullet)$.

(b) The sequence $(V_n)_n$ URMS if and only if $\mathrm{wt}(V) < \infty$ and there is an N s.t. for every $k \leq \mathrm{wt}(V_\bullet)$ and every $\sigma \in S_k$, the character value

$$\chi^{V_n}(\sigma\,(n-k+1\ n-k\ \cdots\ n))$$

stabilizes for $n \geq N$. In this case, a uniform stable range of $V_\bullet$ is given by

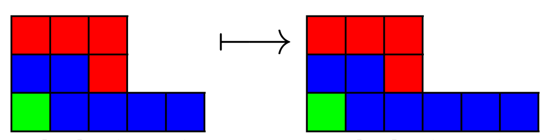
$$n \geq \max\left(2 \cdot \mathrm{wt}(V_\bullet) + 1,\ N\right). \tag{5}$$

Proof idea

The idea is to use change of basis between (2) and (3)

$$c_{\lambda,n} = \sum_{|\mu| \leq |\lambda|} d_{\mu,n} K^{-1}_{\mu[n],\lambda[n]},$$

where $K^{-1}_{\mu[n],\lambda[n]}$ are the **inverse Kostka numbers**, signed sums of special rim hook tableaux of shape $\lambda[n]$ and content $\mu[n]$. Then $K^{-1}_{\mu[n],\lambda[n]} = K^{-1}_{\mu[n+1],\lambda[n+1]}$ for large n because of certain sign-preservig bijective maps



Applications

Diagonal coinvariants

By permuting variables $\sigma \cdot x_i = x_{\sigma(i)}$, $\sigma \cdot y_i = y_{\sigma(i)}$, S_n acts on the graded algebra $BP_n = \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ and so also on

$$DR_n := BP_n / (BP_n)^{S_n}_+.$$

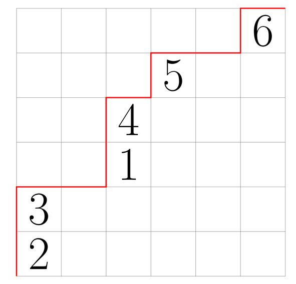
Proposition (Church, Farb, Ellenburg 2014 [2])

The sequence of (i,j) bi-graded components, $[DR_n]_{(i,j)}$ is representation stable with weight $i+j$ and uniform stable range $n \geq 2(i+j)$.

While the Schur expansion of $\mathrm{Frob}([DR_n]_{(i,j)})$ is poorly understood, Carlsson and Mellit [1] proved a formula for the monomial coefficients in terms of **labelled Dyck paths**. We leverage this to get a refinement of the above statement.

Proposition (B. 2025+)

The multiplicity of $\mathbb{S}^{\lambda[n]}$ in $[DR_n]_{(i,j)}$ stabilizes once $n \geq |\lambda| + \max(|\lambda|, i+j)$.



Labeled Dyck path contributing to the coefficient of $m_{(2,2,1,1)}$ in $[DR_6]_{(5,3)}$

Macdonald polynomials

There is a certain S_n-subrepresentation $MD_{\mu[n]} \subset BP_n$,

$$MD_{\mu[n]} := \text{Span of partial derivatives of } \Delta_{\mu[n]}(x_1, \ldots, x_n, y_1, \ldots, y_n),$$

where $\Delta_{\mu[n]}$ is the **bialternant determinant**. The graded Frobenius image

$$\mathrm{Frob}(MD_{\mu[n]}) = \tilde{H}_{\mu[n]}[X; q, t]$$

is the **modified Macdonald polynomial**. The Schur expansion is poorly understood, but there is a monomial formula in terms of $\mu[n]$-fillings due to Haglund, Haiman, and Loehr [3].

Proposition (B. 2025+)

The multiplicity of $\mathbb{S}^{\lambda[n]}$ in $[MD_{\mu[n]}]_{(i,j)}$ stabilizes once $n \geq |\lambda| + \max(|\lambda|, |\mu| + \mu_1 + i)$. The sequence is URMS with stable range $n \geq i + j + \max(i+j, |\mu| + \mu_1 + i)$.

References and acknowledgements

I thank George Seelinger and my advisor, Jim Haglund, for helpful conversations

[1] Carlsson, E., and Mellit, A. A proof of the shuffle conjecture, 2018.

[2] Church, T., Ellenberg, J., and Farb, B. Fi-modules and stability for representations of symmetric groups, 2014.

[3] Haglund, J., Haiman, M., and Loehr, N. A combinatorial model for the macdonald polynomials. *Proceedings of the NAS 101*, 46 (Nov. 2004).

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